THE TAIL OF A POSITIVITY-PRESERVING SEMIGROUP

BY

WILLIAM A. VEECH†

ABSTRACT

With certain assumptions a representation theorem is proved for the elements of $\bigcap_{\sigma \in \Sigma} \sigma S$, where Σ is an abelian semigroup of endomorphisms of a real vector space, and S is a convex antisymmetric cone. Application is made to characterization of nonnegative harmonic functions on bounded Lipschitz domains, of Hausdorff-Stieltjes moment sequences, and of "bilateral Laplace transforms" on locally compact abelian groups, Euclidean motion groups, and non-compact semi-simple Lie groups. Uniqueness of the representation is proved in both the Euclidean motion and the semi-simple cases.

1. Introduction

Let there be given a real vector space V and a cone $S \subseteq V$ which is stable under a certain semigroup Σ of endomorphisms of V. By the *tail* of Σ we shall mean the set $\mathscr{S}$ of vectors v which are common to all the cones σS, $\sigma \in \Sigma$. In what follows we shall make assumptions on V, S, and Σ which will enable us to give a concrete description of $\mathscr{S}$. As an example, let V be the space of all real-valued sequences, $x = (x_0, x_1, \cdots)$, and let S be the cone of nonnegative elements in V. If Σ is the cyclic semigroup generated by T, $(Tx)_n = x_n + x_{n+1}$, then a simple consequence of the main result in Section 2 (or Section 3) is that $\mathscr{S}$ is the set of finite Stieltjes moment sequences

$$x_n = \int_0^\infty \lambda^n \mu(d\lambda) \qquad n = 0, 1, \cdots. \tag{1.1}$$

It is an easy matter to verify directly that every *completely monotone* sequence

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belongs to $\mathscr{S}$, and therefore the classical characterization, due to Hausdorff, of completely monotone sequences as moment sequences (1.1), with μ concentrated on $[0,1]$, also follows (Section 6).

2. An abstract representation theorem

Let V be a real locally convex Hausdorff topological vector space, and let $S \subseteq V$ be a convex cone. We assume $0 \notin S$. Let Σ be an *abelian* semigroup of continuous linear operators on V such that $\Sigma S \subseteq S$.

DEFINITION 2.1. An element $w \in S$ is said to be *divisible* (by Σ) if $w \in \sigma S$ for every $\sigma \in \Sigma$. We refer to the cone $\mathscr{S}$ of divisible vectors as the *tail* of Σ.

If $w \in S$ is an eigenvector for Σ ($\sigma w = \lambda(\sigma)w, \sigma \in \Sigma$), then obviously $w \in \mathscr{S}$. The object of the discussion to follow is to delineate a set of assumptions on V, S, and Σ which will imply that $\mathscr{S}$ can be synthesized from the eigenvectors it contains.

DEFINITION 2.2. We say (Σ, S) has the *compactness property* if for every $\sigma \in \Sigma$ and $w \in \mathscr{S}$ the set

$$K(\sigma, w) = \text{closure } \sigma\{\sigma^{-2}w \cap S\}$$

is compact.

Notice that $K(\sigma, w) \subseteq S$ because S is closed, and $\sigma K(\sigma, w) = \{w\}$ because σ is continuous. In many instances $\sigma^{-1}w \cap S$ is already compact, but for certain applications the weaker assumption (2.2) must suffice.

PROPOSITION 2.3. *If (Σ, S) has the compactness property, then $\sigma\mathscr{S} = \mathscr{S}$ for all $\sigma \in \Sigma$.*

PROOF. Σ is assumed to be abelian, and therefore $\sigma\mathscr{S} \subseteq \mathscr{S}$ for all $\sigma \in \Sigma$. Fixing $w \in \mathscr{S}$ and $\sigma \in \Sigma$ we must find $v \in \mathscr{S}$ such that $\sigma v = w$. By continuity and commutativity we have that $\gamma K(\gamma\sigma, w)$ is a compact subset of $K(\sigma, w)$ for every $\gamma \in \Sigma$. Therefore, if we can prove that the sets $\gamma K(\gamma\sigma, w)$, $\gamma \in \Sigma$, have the finite intersection property, it will follow that they have a vector v in common, and certainly $v \in \mathscr{S}$. To prove the finite intersection property let $n \geqq 1$, and let $\gamma_1, \cdots, \gamma_n \in \Sigma$ be given. Choose $u \in S$ so that $\gamma_1^2\gamma_2^2 \cdots \gamma_n^2\sigma^2 u = w$. This we may do because $w \in \mathscr{S}$. Now for $1 \leqq i \leqq n$ define $u_i = \gamma_1^2\gamma_2^2 \cdots \gamma_{i-1}^2\gamma_i\gamma_{i+1}^2 \cdots \gamma_n^2\sigma u$. By commutativity $\gamma_i u_i = \gamma_j u_j$, $1 \leqq i, j \leqq n$, and $\gamma_i u_i \in \gamma_i K(\gamma_i\sigma, w)$. Thus,

$$\bigcap_{i=1}^{n} \gamma_i K(\gamma_i\sigma, w) \neq \varnothing,$$

and the finite intersection property is established. The proposition is proved.

Recall that a cone $\mathscr{S}$ is *well capped* if for each $w \in \mathscr{S}$ there is an extremal cone $\mathscr{S}_w \subseteq \mathscr{S}$ (meaning that if $u_1, u_2 \in \mathscr{S}$ and $u_1 + u_2 \in \mathscr{S}_w$, then $u_1, u_2 \in \mathscr{S}_w$) with $w \in \mathscr{S}_w$ and a positive linear functional ϕ on $\mathscr{S}_w$ such that $\{v \in \mathscr{S}_w \mid \phi(v) \leqq 1\}$ is compact when 0 is adjoined.

In our applications $\{0\} \cup S$ has the property that every compact subset is metrizable. For this reason, and in order to avoid the pathologies encountered in nonmetrizable Choquet theory, we assume in all that follows that $\{0\} \cup S$ has only metrizable compacta.

Given that $\mathscr{S}$ is well capped, we associate to $w \in \mathscr{S}$ a pair $\mathscr{S}_w$ and ϕ as in the above. Define $K_w = \{v \in \mathscr{S}_w \mid \phi(v) \leqq 1\}$. Since $w/\phi(w) \in K_w$, and since $\{0\} \cup K_w$ is a metrizable, compact convex set, there exists by the Choquet theorem a Borel measure η on the Borel (even G_δ) set L_w of extreme points of K_w such that

$$w = \int_{Lw} e\eta(de). \tag{2.4}$$

The representation (2.4) holds in the weak sense. That is, if $\psi \in V^*$ (the dual space to V), then

$$\psi(w) = \int_{Lw} \psi(e)\eta(de). \tag{2.5}$$

Now we will assume there exists $\psi \in V^*$ which is strictly positive on $\mathscr{S}$. In this case the set $\mathscr{S}_1 = \{w \in \mathscr{S} \mid \psi(w) = 1\}$ is a *base* for $\mathscr{S}$. If we map L_w above into $\mathscr{S}_1$ by $e \to e/\psi(e)$, then (2.4)–(2.5) can be viewed as a representation

$$w = \int_{\mathscr{S}_1} e\xi(de). \tag{2.6}$$

Moreover, every extreme point of K_w is an extremal of the cone $\mathscr{S}$. Indeed, if $e \in L_w$, and if $e = e_1 + e_2$, $e_1, e_2 \in \mathscr{S}$, then $e_1, e_2 \in \mathscr{S}_w$ because $\mathscr{S}_w$ is extremal. Clearly $\phi(e) = 1$ and so $e = \phi(e_1)(e_1/\phi(e_1)) + \phi(e_2)(e_2/\phi(e_2))$ is a convex combination of $e_i/\phi(e_i)$, $i = 1, 2$. Since e is extreme, $e_i = \phi(e_i)e$, and e is an extremal of $\mathscr{S}$. Let E be the set of extremals of $\mathscr{S}$ which belong to $\mathscr{S}_1$. From (2.6) and the preceding discussion we obtain the representation

$$w = \int_E e\xi(de)$$

where ξ is a regular Borel measure on E. The mass of ξ is $\psi(w)$.

REMARK. 2.7. Denote by σ^* the adjoint to the transformation $\sigma \in \Sigma$. For each $\tau \in V^*$, (2.6) implies

$$\begin{aligned}\tau(\sigma w) &= (\sigma^*\tau)(w)\\ &= \int_E \sigma^*\tau(e)\xi(de)\\ &= \int_E \tau(\sigma e)\xi(de)\end{aligned}$$

and therefore

$$\sigma w = \int_E \sigma e \xi(de). \tag{2.8}$$

DEFINITION 2.9. We say that Σ *decays slowly* if for each $\sigma \in \Sigma$ there is an $a > 0$ such that $\sigma^2 > a\sigma$ on S. *Greater than* is in the sense of the natural order induced by S. If $v \in S$, then $\sigma^2 v - a\sigma v \in S$.

PROPOSITION 2.10. *Assume* (Σ, S) *has the compactness property and* Σ *decays slowly. Then every extremal of* $\mathscr{S}$ *is an eigenvector for* Σ.

PROOF. Let $e \in \mathscr{S}$ be an extremal. Given $\sigma \in \Sigma$ there exists by Proposition 2.3 an element $v \in \mathscr{S}$ such that $\sigma^2 v = e$. Let $a = a(\sigma) > 0$ be as in Definition 2.9. We claim $e - a\sigma v \in \mathscr{S}$. For if $\gamma \in \Sigma$, and if $u \in S$ is such that $\gamma u = v$, then $\gamma(\sigma^2 u - a\sigma u) = \sigma^2 v - a\sigma v = e - a\sigma v$, and $\sigma^2 u - a\sigma u \in S$ by our choice of a. Since e is an extremal, there exists $t > 0$ such that $a\sigma v = te$. Apply σ to both sides and divide by t to find $\sigma e = (a/t)e$. Since $\sigma \in \Sigma$ is arbitrary, the proposition is proved.

If the assumptions of Proposition 2.10 are in force, then every extremal $e \in \mathscr{S}$ determines a character $\lambda: \Sigma \to \mathbb{R}^+$, $\mathbb{R}^+$ the group of positive reals under multiplication, by $\sigma e = \lambda(\sigma)e$. We will speak of the eigenvector e as *belonging to* λ. We denote by $\mathscr{S}(\lambda)$ the set of vectors $w \in S$ (not necessarily extremals) which belong to λ. Of course, $\mathscr{S}(\lambda) \subseteq \mathscr{S}$, and $\mathscr{S}(\lambda)$ is a closed subcone of $\mathscr{S}$. Let $E(\lambda)$ be the extremal elements of $\mathscr{S}(\lambda)$ which belong to $\mathscr{S}_1$ (that is, $\psi(e) = 1$, where ψ is the distinguished functional which is positive on $\mathscr{S}$). Proposition 2.10 implies $E \subseteq \cup_\lambda E(\lambda)$. We will now prove the reverse inclusion.

PROPOSITION 2.11. *Under the hypotheses of Proposition* 2.10, $E = \cup_\lambda E(\lambda)$.

PROOF. Given λ and $w \in E(\lambda)$ it is to be proved that $w \in E$. If we replace each $\sigma \in \Sigma$ by $\sigma/\lambda(\sigma)$ we may assume $\lambda = 1$. Since $w \in \mathscr{S}$, there exists a representation

$$w = \int_E e\mu(de).$$

If we write $\sigma e = \lambda_e(\sigma)$, then (2.8) implies for every $\sigma \in \Sigma$ and $n \geqq 1$

$$w = \sigma^n w = \int_E \lambda_e(\sigma)^n e\mu(de)$$

and

$$1 = \psi(w) = \psi(\sigma^n w) = \int_E \lambda_e(\sigma)^n \mu(de).$$

It follows that $\mu\{e \mid \lambda_e(\sigma) \neq 1\} = 0$. Now more must be said because Σ may be uncountable. Let $A \subseteq E$ be any compact subset of E. A is metrizable, and therefore there exists a countable set $\sigma_1, \sigma_2, \cdots \in \Sigma$ such that if $F_n(e) = \lambda_e(\sigma_n)$, $e \in K$, then $\{F_n\}$ is uniformly dense (on A) in the set $\{F_\sigma(e) = \lambda_e(\sigma) \mid \sigma \in \Sigma\}$. Now if $A_1 = \{e \in A \mid F_n(e) = 1, \text{ all } n\}$, then $A_1 = \{e \in A \mid F_\sigma(e) = 1, \text{ all } \sigma \in \Sigma\}$. Since $\mu(A_1^c \cap A) = 0$, and since μ is regular, we conclude that $\mu\{e \in E \mid F_\sigma(e) = 1, \text{ all } \sigma \in \Sigma\} = 1$. That is, μ is concentrated on $E(1)$. Since w is an extremal element of $E(1)$, μ must be concentrated at w. Therefore, $w \in E$. The proposition is proved.

THEOREM 2.12. *Let Σ be an abelian semigroup of endomorphisms of a real, Hausdorff, locally convex, topological vector space V and let S be a closed Σ-stable cone in V. Let $\mathscr{S} = \bigcap_{\sigma \in \Sigma} \sigma S$ be the tail of Σ. If*

(i) *the compact sets in $\{0\} \cup S$ are metrizable,*
(ii) *(Σ, S) has the compactness property,*
(iii) *$\mathscr{S}$ is well-capped and there exists $\psi \in V^*$ which is positive on $\mathscr{S}$, and*
(iv) *Σ decays slowly,*

then every $w \in \mathscr{S}$ admits a representation

$$w = \int_E e\mu(de) \tag{2.13}$$

in which μ is a regular Borel measure on the set of extremals of $\mathscr{S}$ such that $\psi(e) = 1$. Moreover $E = \bigcup_\lambda E(\lambda)$, where for each positive character λ on Σ, $E(\lambda)$ is the set of extremals e of the cone *$\mathscr{S}(\lambda)$ of vectors belonging to λ such that $\psi(e) = 1$.*

REMARK 2.14. Assume S has the property that every weakly Cauchy sequence is weakly convergent. Then for any Borel measure μ on E which has compact support (2.13) exists and defines an element of $\mathscr{S}$. If μ is not necessarily compactly supported but has the property that for all $\eta \in V^*$, $\eta(\cdot)$ is μ-integrable, then we claim (2.13) exists and defines an element of $\mathscr{S}$. For by regularity there is a sequence $K_n \subseteq K_{n+1}$ of compact subsets of E such that μ is supported on $\cup_{n=1}^{\infty} K_n$. Let w_n be the resultant of $\mu_n = \chi_{K_n}\mu$ (χ = characteristic function). w_n exists because the weak sequential completeness assumption implies K_n has a compact closed convex hull. If $\eta \in V^*$, then because $\eta(\cdot)$ is μ integrable, $\eta(w_n)$ is convergent to ${}_E\int \eta(e)\mu(de)$. It follows there exists $w \in S$ such that $w_n \to w$ weakly, and w is represented by (2.13). Because Σ is slowly decaying it is true for any fixed $\sigma \in \Sigma$ that $\lambda_0(\sigma) \geqq a(\sigma) > 0$ on E. Therefore, if we apply the above argument with $\mu(de)$ replaced by $\mu(de/\lambda_e(\sigma)$, (2.13) defines an element $v \in S$ such that $\sigma v = w$. It follows $w \in \mathscr{S}$.

REMARK 2.15. Assumption (iii) in the statement of Theorem 2.12 can be weakened slightly to read: $(c')\mathscr{S}$ is well capped and there exists $\psi \in V^*$ such that $\psi(e) > 0$ for every extremal of $\mathscr{S}$. For then, by the integral representation (2.6), $\psi(w) > 0$ for *every* $w \in \mathscr{S}$.

3. Semigroups of integral operators

Let Ω be a second countable, locally compact, metric space, and let K be a real-valued function on $\Omega \times \Omega$. If $A \subseteq \Omega$, denote by $M = M(K, A)$ the number (or ∞)

$$M(K,A) = \sup_{(x,y) \in A \times \Omega} K(x,y)$$

and by $C = C(K, A)$ the *set*

$$C(K,A) = \bigcup_{x \in A} \text{closure } \{y \mid K(x,y) > 0\}.$$

Now $\mathscr{K}$ will be the set of non-negative Borel functions, K on $\Omega \times \Omega$ such that for every compact set $A \subseteq \Omega$, $M(K, A)$ is finite and $C(K, A)$ has compact closure. Fixed for the discussion is a locally finite Borel measure ν on Ω. Using ν, we introduce a binary operation on $\mathscr{K}$, defining $K_1 \circ K_2$ for $K_1, K_2 \in \mathscr{K}$ by

$$K_1 \circ K_2(x,y) = \int_{\Omega} K_1(x,z)K_2(z,y)\nu(dz).$$

$\mathscr{K}$ is closed under this operation as an elementary calculation shows.

DEFINITION 3.1. An element $K \in \mathscr{K}$ is *admissible* if for every compact set $A \subseteq \Omega$ there is a number $\alpha = \alpha(K, A) > 0$ such that for all $x \in A$, $K \circ K(x, \cdot) \geqq \alpha K(x, \cdot)$. If α can be chosen independently of A, K is *strongly admissible.*

A set $\Sigma \subseteq \mathscr{K}$ is admissible (strongly admissible) if each of its elements is admissible (strongly admissible).

Let $\mathscr{L}^+_{\text{loc}}$ be the space of equivalence classes of locally ν-integrable non-negative functions on Ω. We let $\mathscr{K}$ act on $\mathscr{L}^+_{\text{loc}}$ by $f \to Kf$, where

$$Kf(x) = \int_\Omega K(x,y)f(y)\nu(dy). \tag{3.2}$$

Lemma 3.3. *Let $K \in \mathscr{K}$ be admissible, and suppose $f = Kg$, $g \in \mathscr{L}^+_{\text{loc}}$. Then $f \in \mathscr{L}^\infty_{\text{loc}}$, the space of locally bounded ν-measurable functions on Ω. In fact, for any compact set $A \subseteq \Omega$ there is a constant $N = N(K,A) < \infty$ such that*

$$\operatorname*{ess\,sup}_{x \in A} f(x) \leqq N(K,A)\int_{C(K,A)} f(y)\nu(dy). \tag{3.4}$$

Proof. Let A be compact, and let $M = M(K,A)$ and $C = C(K,A)$ be as in the introductory paragraph to this section. Choose $\alpha = \alpha(K,A)$ as in the definition of admissibility. For almost all $x \in A$ we have

$$\begin{aligned} f(x) &= Kg(x) \\ &\leqq \frac{1}{\alpha}K \circ Kg(x) \\ &= \frac{1}{\alpha}Kf(x) \\ &\leqq \frac{M}{\alpha}\int_C f(y)\nu(dy). \end{aligned}$$

The lemma is proved.

Lemma 3.5. *Suppose $f = K \circ Kg$ for some $g \in \mathscr{L}^+_{\text{loc}}$ and admissible $K \in \mathscr{K}$. Set $h = Kg$. For every compact set $A \subseteq \Omega$ there is a constant $R = R(K,A,f) < \infty$ such that*

$$\operatorname*{ess\,sup}_{x \in A} h(x) \leqq R. \tag{3.6}$$

Proof. Given K admissible and A compact choose $\alpha = \alpha(K,A)$ by the definition of admissibility. If f and g are as in the statement of the lemma, we have for almost all x, $f(x) = K \circ Kg(x) \geqq \alpha Kg(x) = \alpha h(x)$. Therefore, by Lemma 3.3,

$$\text{(3.7)} \qquad \operatorname*{ess\,sup}_{x \in A} h(x) \leqq \frac{1}{\alpha} \sup_{x \in A} f(x) \leqq \frac{N(K,A)}{\alpha} \int_{C(K,A)} f(y)\nu(dy).$$

Setting $R(K, A, f)$ equal to the right-hand side of (3.7), the lemma is proved.

In what follows, V is the space $\mathscr{L}^\infty_{\text{loc}}$ and S is the cone of non-negative elements of V, excluding 0. Because $Kg \in S$ when $g \in \mathscr{L}^+_{\text{loc}}$ and $K \in \mathscr{K}$, the notion of divisibility based on solvability of $Kg = f$ for $g \in \mathscr{L}^+_{\text{loc}}$ coincides with the notion of divisibility when g is required to belong to S.

V is the dual space of the space of ν-integrable compactly supported functions on Ω (with the direct limit topology), and we endow V with the weak-* topology. Any subset of V for which there exists a local (essential) uniform bound will, by the Alaoglu theorem, be relatively compact in this topology. Since Ω is separable and ν is locally finite, V has metrizable compacta.

LEMMA 3.8. *For each $K \in \mathscr{K}$ the operator $f \to Kf$ is an endomorphism of V.*

PROOF. If h is compactly supported and ν-integrable, then so is $hK(y) = \int K(x, y)h(x)\nu(dx)$. Continuity therefore follows from the Fubini theorem and the definition of the topology on V.

LEMMA 3.9. *If Λ is a locally uniformly bounded subset of S, and if $K \in \mathscr{K}$ is admissible, then $K\{g \in S \mid K \circ Kg \in \Lambda\}$ is relatively compact.*

PROOF. Lemma 3.5 (see (3.7)) gives a local uniform bound for the elements of the set in question. Therefore, by an earlier remark, it has compact closure in S.

LEMMA 3.10. *Let Σ be an admissible subsemigroup of $\mathscr{K}$, and let $\mathscr{S} = \cap_{K \in \Sigma} KS$. Then $\mathscr{S}$ is a well-capped cone.*

PROOF. Given $f \in \mathscr{S}$ we use the second countability of Ω to find an everywhere positive continuous function h on Ω such that $(f, h) < \infty$, where $(f,h) = {}_\Omega\int f(x)h(x)\nu(dx)$. Let W be the cone of functions $g \in S$ such that $(g,h) < \infty$. Notice that W is extremal. For every compact set $C \subseteq \Omega$ we define $v = v(C) = \min_{x \in C} h(x)$. $v > 0$ because h is continuous and positive. For such a set C we have the inequality

$$\text{(3.11)} \qquad \int_C f(x)\nu(dx) \leqq \frac{(f, h)}{v(C)}.$$

Specializing (3.11) to the case of $C = C(K, A)$, A compact, K admissible, (3.4) and (3.11) combine to imply

$$\operatorname*{ess\,sup}_{x \in A} f(x) \leqq N(K, A) \frac{(f, h)}{v(C)}. \tag{3.12}$$

If $W_1 = \{\phi \in \mathscr{S} \mid (\phi, h) \leqq 1\}$, (3.12) implies W_1 has compact closure in S. In fact, it will be seen that W_1 is already closed and therore itself compact.

Suppose $\{f_n\}$ is a net in W_1 which is convergent to $f \in S$. If A is any compact subset of Ω, $(f, \chi_A h) = \lim_n (f_n, \chi_A h) \leqq 1$, and since A is arbitrary, $(f, h) \leqq 1$. Fixing $K \in \Sigma$ we choose for every n, $g_n \in S$ such that $K \circ Kg_n = f_n$. By (3.12), $\{f_n\}$ is locally uniformly bounded, and therefore by Lemma 3.9 we may choose a subnet if necessary and assume Kg_n is convergent to some limit h. Then $Kh = f$ by continuity. Since K is arbitrary, $f \in \mathscr{S}$.

Collecting results so far, we have the following proposition.

PROPOSITION 3.13. *Let Σ be a strongly admissible subsemigroup of $\mathscr{K}$, and let $\mathscr{S} = \cap KS$. Then*

(i) *(Σ, S) has the compactness property,*

(ii) *Σ is slowly decaying, and*

(iii) *$\mathscr{S}$ is well capped.*

PROOF. Statements (i)–(iii) follow respectively from Lemma 3.9, strong admissibility, and Lemma 3.10.

In what follows we shall say $K \in \mathscr{K}$ *increases support* if for each $x \in \Omega$ there exists a number $\delta > 0$ and a neighborhood U of x such that $K(x, \cdot) \geqq \delta$ on U. If K is support increasing, then $f \geqq 0$ and $Kf = 0$ implies $f = 0$. Thus, if K increases support, S is stable under K. We say a set $\Sigma \subseteq \mathscr{K}$ is *support increasing* if each of its elements is.

THEOREM 3.14. *Let v be a locally finite Borel measure on the separable locally compact metric space Ω, and let Σ be a strongly admissible support increasing, abelian subsemigroup of $\mathscr{K}$. Assume there is a compact set $B \subseteq \Omega$ such that $\sup_{K \in \Sigma}(Kf, \chi_B) > 0$ for every $f \in S$. If $\mathscr{S} = \cap_{K \in \Sigma} KS$ is the tail of Σ, then every $f \in \mathscr{S}$ admits a representation*

$$f = \int_E e\mu(de) \tag{3.15}$$

where μ is a Borel measure on the set $E = \cup_\lambda E_\lambda$, the union taken over all positive *characters λ on Σ, and for each λ, E_λ being the set of $e \in S$, $(e, \chi_B) = 1$, which are extremals of the cone $\mathscr{S}(\lambda)$ of elements of S which belong to λ.*

PROOF. Σ is strongly admissible, and therefore each extremal e of $\mathscr{S}$ is an eigenfunction for Σ. Let e belong to λ. Then $(Ke, \chi_B) = \lambda_e(K)(e, \chi_B)$, $K \in \Sigma$, and therefore $(e, \chi_B) > 0$. Since $\mathscr{S}$ is well capped, (3.15) follows from Remark 2.15 and Theorem 2.12.

REMARK 3.16. With notation as above, suppose $f \in \mathscr{S}$ and choose any $K \in \Sigma$ and $g \in \mathscr{S}$ such that $Kg = f$. Define a (Borel) function f_0 on Ω by $f_0(x) = Kg(x) = (g, K(x, \cdot))$. We claim f_0 is divisible in the sense that if $K_0 \in \Sigma$, there exists a Borel function $g_0 \geqq 0$ on Ω such that $f_0(x) = K_0 g_0(x)$ for all x. Indeed, find $h \in S$ such that $K_0 h = g$, and define $g_0(x) = Kh(x)$. That $K_0 g_0 = f_0$ follows from the commutativity of Σ. Since K_0 is arbitrary, f_0 is divisible as a point function on Ω. Moreover, if we represent g above by (3.15) then, since for each $\chi \in \Omega$, $K(x, \cdot) \in V^*$, we have

$$
(3.17) \qquad f_0(x) = \int (K(x, \cdot), e) u(de) = \int e_0(x) \mu(de)
$$

holding for all x, where for each e_0, $e_0(\cdot)$ is a point eigenfunction for K.

If f is a divisible point function for Σ, then $[f]$, the element of S corresponding to f is evidently an element of $\mathscr{S}$. The representation (3.15) for $[f]$ will at least hold almost everywhere for f_0. With a bit more care it is possible to obtain a representation for f_0 which holds for all x. The key is a modification of the argument in Proposition 2.3 which yields $g \in \mathscr{S}$ for a given $K \in \Sigma$ such that $f_0(x) = (K(x, \cdot), g)$, $x \in \Omega$. The modification turns on the fact that for fixed K $\{g \in S \mid (K(x, \cdot), g) = f_0(x),\ x \in \Omega\}$ is closed (and nonempty), and therefore the same compactness argument can be used.

For a final remark, suppose $e \in \mathscr{S}$ is an extremal, and fix $K \in \Sigma$. Define a Borel function e_K on Ω by $e_K(x) = ((K(x, \cdot), e)/\lambda_e(K))$. Then $e_K(\cdot)$ is equivalent to e (belongs to the equivalence class e represents), and moreover, because Σ is abelian and $K_0 e = \lambda_e(K_0)e$, $K_0 \in \Sigma$ we have $K_0 e_K(x) = \lambda_e(K_0) e_K(x)$, $x \in \Omega$. The left side of the last equality depends only upon e, and therefore $e_K(x)$ is the unique element of e which is a point-eigenfunction for Σ. In case the representation (3.15) is unique for $f \in \mathscr{S}$, it follows readily that each $f \in \mathscr{S}$ contains a unique Borel $f_0(x)$ which is divisible as a point function. We do not know if this is true in general.

4. Convolution operators

Let G be a fixed second countable, Hausdorff, locally compact topological group, and let ν be a left Haar measure on G. G will play the role in what follows of the space Ω of Section 3. If ϕ is a non-negative, bounded, measurable, compactly supported function on G which is bounded away from 0 on some neighborhood of $e \in G$ (e = identity), we set up a kernel K on $G \times G$ by $K(x, y) = \phi(xy^{-1})$. $\mathscr{K}$ denotes the space of all such kernels, and we write $\phi \in \mathscr{K}$ rather than $K \in \mathscr{K}$. Notice that if $K_i(x, y) = \phi_i(xy^{-1})$, $i = 1, 2$, then $K_1 \circ K_2(x, y) = \phi_1 * \phi_2(xy^{-1})$, where $*$ denotes convolution.

If $\phi \in \mathscr{K}$, then because ϕ is bounded away from 0 on a neighborhood of e it will be the case that $\phi * \phi(x) > 0$ whenever there does not exist a neighborhood of x on which ϕ vanishes essentially. Since $\phi * \phi$ is continuous, and since ϕ is bounded, there will exist a number $\alpha > 0$ such that $\phi * \phi \geqq \alpha\phi$. It follows $\mathscr{K}$ is strongly admissible.

In what follows we take Σ to be a fixed abelian sub-semigroup of $\mathscr{K}$ with the property that $\cup_{\phi \in \Sigma} \{x \mid \phi(x) > 0\}^\circ = G$, where $^\circ$ denotes interior.

LEMMA 4.1. *With notations and asumptions as above, there exists for every compact set $A \subseteq G$ an element $\phi \in \Sigma$ which is bounded away from 0 on A.*

PROOF. For every $x \in A$ choose a continuous $\phi_x \in \Sigma$ with $\phi_x(x) > 0$, and let $U_x = \{y \mid \phi_x(y) > 0\}$. Then $\{U_x \mid x \in A\}$ is an open cover of A. Choose $x_1, \cdots, x_n$ so that $A \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$, and define $\phi = \phi_{x_1} * \cdots * \phi_{x_n}$. That $\phi > 0$ on A follows because each $\phi \in \mathscr{K}$ is support increasing.

One immediate consequence of the lemma is that if $f \in S$ (equivalence classes of non-negative, locally bounded, ν-measurable functions $\not\equiv 0$), and if B is any compact neighborhood of e, then $\sup_{\phi \in \Sigma}(\phi * f, \chi_B) > 0$. Fixing such a neighborhood and letting E be the set of extremal eigenfunctions e for Σ such that $(e, \chi_B) = 1$, it follows from Theorem 3.14 that every $f \in \mathscr{S}$ has a representation

$$f(x) = \int_E e(x)\mu(de) \tag{4.2}$$

holding almost everywhere. Moreover if $f(\cdot)$ is a divisible point function, then (4.2) can be chosen so as to hold everywhere. With various assumptions on G it is possible to give more concreteness to (4.2) by identifying the eigenfunctions for Σ.

LEMMA 4.3. *With notation and assumptions as above, suppose G contains a closed normal abelian subgroup L and a compact subgroup K such that $G = LK$, in the sense that the map $(l, k) \to lk$ is a homeomorphism from $L \times K$ into G. If f is an extremal eigenfunction for Σ, there exists a continuous positive character c on L such that if $x = lk$, $l \in L$, $k \in K$, then $f(x) = c(k^{-1}lk)f(k)$.*

PROOF. Fix any $\phi \in \Sigma$, and let λ be such that $\phi * f = \lambda f$. If $x = lk$, and if $t \in L$, define $t' \in L$ by $t' = ktk^{-1}$. We have

$$\lambda f(xt) = \int_G \phi(xty^{-1})f(y)v(dy)$$

$$= \int_G \phi(t'xy^{-1})f(y)v(dy)$$

$$= \int_G \phi(t'y^{-1})f(yx)v(dy).$$

If $\psi \in \Sigma$ and η are such that $\psi * f = \eta f$, then

$$\eta f(x) = \int_G \psi(y^{-1})f(yx)v(dy).$$

Let $A_t = \{t' = ktk^{-1} \mid k \in K\}$. A_t is compact because K is as is A = closure $\{\tau^{-1}y \mid \tau \in A_t, \phi(y) > 0\}$. Choose ψ to be bounded away from 0 on A. Then for some $\alpha > 0$, $\psi(y) \geqq \alpha\phi(t'y)$, $t' \in A_t$, $y \in G$. We conclude $\eta f(x) \geqq \alpha\lambda f(xt)$, $x \in G$, and since f is an extremal eigenfunction and both $f(x)$ and $f(xt)$ belong to the same eigenvalue, $f(xt) = c(t)f(x)$. Writing $x = lk = kk^{-1}lk$, we see that $f(x) = c(k^{-1}lk)f(k)$. Clearly, c is a character. The lemma is proved.

The statements and proofs of the next two lemmas are based on [6, Lemmas 8.1 and 10.1]. L^+ is the set of positive continuous characters on L.

LEMMA 4.4. *Let $\phi \in \mathscr{K}$ be continuous, and let $c \in L^+$. There exists a continuous function p on K such that if $f(x) = c(k^{-1}lk)p(k)$, $x = lk$, $l \in L$, $k \in K$, then f is an eigenfunction for ϕ.*

PROOF. Given c we will solve for p. Thus write $f(lk) = c(k^{-1}lk)p(k)$, and let $y = lk$, $x = l_1k_1$ in what follows.

$$\begin{aligned}\phi * f(x) &= \int \phi(l_1 k_1 k^{-1} l^{-1}) f(lk) \nu(d(lk)) \\ &= \int \phi(k^{-1} l^{-1}) f(lk l_1 k_1) \nu(d(lk)) \\ &= \int \phi(k^{-1} l^{-1}) f(lk l_1 k^{-1} k_1) \nu(d(lk)) \\ &= \int \phi(k^{-1} l^{-1}) c(k_1^{-1} k^{-1} lk l_1 k^{-1} k k_1) p(k k_1) \nu(d(lk)) \\ &= \int \phi(k^{-1} l^{-1}) c(k_1^{-1} k^{-1} lk k_1) c(k_1^{-1} l_1 k_1) p(k k_1) \nu(d(lk)) \\ &= \left\{ \int \phi(k^{-1} l^{-1}) c(k_1^{-1} k^{-1} lk k_1) \frac{p(k k_1)}{p(k_1)} \nu(d(lk)) \right\} f(x).\end{aligned}$$

The problem now is to find p so that the expression in braces is independent of k_1. What is the same thing, it is necessary to find an eigenfunction $p > 0$ for the operator

$$Tp(k_1) = \int \phi(y^{-1}) \sigma(y, k_1) p(k k_1) \nu(d(lk)) \tag{4.5}$$

in which $\sigma(y, k_1) = c(k_1^{-1} k^{-1} lk k_1)$, $y = lk$. We note that for $k_2 \in K$, $\sigma(yk_2, k_1) = c(k_1^{-1} k_2^{-1} k^{-1} lk k_2 k_1) = \sigma(y, k_2 k_1)$. Therefore if we let $w = yk_1$ in (4.5), $w = lk$,

$$Tp(k_1) = \int \phi(k_1 w^{-1}) \sigma(w, e) p(k) \nu(d(lk)).$$

Let A be a compact set which contains all $w \in G$ such that $\phi(k_1 w^{-1}) > 0$ for some $k_1 \in K$. If $I(p) = {}_K\int p(k) dk$ (dk = normalized Haar measure on K), it is readily checked that because $\phi(e)\sigma(e,e) > 0$, $I(Tp) \geqq \beta I(p)$ for some constant $\beta > 0$. Also, if A is a compact set which contains all $w \in G$ such that $\phi(k_1 w^{-1}) > 0$ for some $k_1 \in K$,

$$\int_A \sigma(w, e) p(k) \nu(d(lk)) \leqq \gamma I(p)$$

for some $\gamma < \infty$ independent of p. It follows that if $k_1, k_2 \in K$, then

$$Tp(k_1) - Tp(k_2) \big| \leqq \gamma I(p) \sup_w \big| \phi(k_1 w^{-1}) - \phi(k_2 w^{-1}) \big| .$$

Putting these facts together, if we define $\Lambda = \{p \mid I(p) = 1\}$ and $\alpha(p)$, $p \in \Lambda$, so that $T_0 \Lambda \subseteq \Lambda$, where $T_0 p = \alpha(p) Tp$, then $T_0 \Lambda$ is a precompact set in $C(K)$.

Then $\Lambda_1 = \text{closure}\,\text{co}\, T_0\Lambda$ is compact in Λ, and $T_0\Lambda_1 \subseteq \Lambda_1$. By the Schauder-Tychonoff theorem, there exists $p \in \Lambda_1$, $T_0 p = p$. Then $Tp = \alpha(p)^{-1}p$. As remarked above this implies the conclusion of the lemma.

REMARK. In the next lemma we shall use the well-known fact that if dl and dk are Haar measures on L and K respectively, then the product measure $dldk$ on $L \times K$ projects under the homeomorphism $(l, k) \to lk$ onto a Haar measure for G.

LEMMA 4.6. *Let Σ be as above, and let $c \in L^+$. There exists at most one eigenfunction f for Σ with $f(e) = 1$ and $f(lk) = c(k^{-1}lk)f(k)$.*

PROOF. Let f_1, f_2 be two such eigenfunctions. We will first observe there is one character on Σ to which both f_1 and f_2 belong. First, since neither f_1 nor f_2 can vanish, there exists a constant M, $0 < M < \infty$, such that $M^{-1}f_1(k) \leqq f_2(k) \leqq Mf_1(k)$, $k \in K$. Since $f_i(lk) = c(k^{-1}lk)f_i(k)$, $l \in L$, $k \in K$, we have also $M^{-1}f_1(x) \leqq f_2(x) \leqq Mf_1(x)$, $x \in G$. Now if $\phi \in \Sigma$, $\phi * f_i = \lambda_i f_i, i = 1, 2$, these inequalities imply for all $n \geqq 1$, $M^{-1}\lambda_1^n f_1(x) \leqq \lambda_2^n f_2(x) \leqq M\lambda_1^n f_1(x)$. Therefore, $\lambda_1 = \lambda_2$. Now for $\phi \in \Sigma$ and $x \in G$ we have

$$\frac{f_2(x)}{f_1(x)} = \frac{\int \phi(g^{-1})f_2(gx)\nu(dg)}{\int \phi(g^{-1})f_1(gx)\nu(dg)}$$

$$= \frac{\int \phi(g^{-1})f_1(gx)(f_2(gx)/f_1(gx))\nu(dg)}{\int \phi(g^{-1})f_1(gx)\nu(dg)}. \tag{4.7}$$

If dl is a Haar measure on L, and if as before dk is normalized Haar measure on K, then $dldk$ is a Haar measure on G. Define $F(\cdot, \cdot)$ on $K \times K$ by

$$F(k_1, k_2) = \int_L \phi(k_1^{-1}\, l^{-1})f_1(lk_1k_2)dl.$$

If $\phi > 0$ on K, then

$$\theta_{k_2}(k_1) = \frac{F(k_1, k_2)}{\int_K F(k, k_2)dk}$$

is the density of a probability measure on K which assigns positive measure to every open set. We have from (4.7) and the fact $f_2(lk)/f_1(lk) = f_2(k)/f_1(k)$,

$$\frac{f_2(k_2)}{f_1(k_2)} = \int_K \frac{f_2(k)}{f_1(k)}\theta_{k_2}(k)dk.$$

If k_2 is chosen so that the left side is small as possible, the integrand $f_2(k)/f_1(k)$ must almost everywhere be equal to this smallest value. Since f_1, f_2 are continuous, $f_1 = \lambda f_2$. Finally, $f_i(e) = 1$, $i = 1,2$, so $\lambda = 1$. The lemma is proved.

LEMMA 4.8. *With notation and assumptions as above, let $c \in L^+$. There exists a unique eigenfunction f for Σ such that $f(e) = 1$ and $f(lk) = c(k^{-1}lk)f(k)$.*

PROOF. Fix a $\phi \in \Sigma$ which is continuous on G and positive on I. There exists a unique such f for ϕ by Lemmas 4.4 and 4.6. If $\psi \in \Sigma$, then because Σ is abelian, $\psi * f$ is also an eigenfunction for ϕ. Since $\psi * f(xt) = c(t)\psi * f(x)$, $t \in L$, uniqueness tells us $\psi * f(x) = ((\psi * f)(e))f(x)$. The lemma is proved.

THEOREM 4.9. *Let G be a second countable locally compact group containing a closed normal abelian subgroup L and a compact subgroup K such that $G = LK$ as above. Let v be a Haar measure on G. Assume Σ is an abelian semigroup, under convolution of non-negative, bounded, measurable, compactly supported functions each of which is bounded away from 0 on some neighborhood of e and such that $\cup_{\phi \in \Sigma}\{x \mid \phi(x) > 0\}^0 = G$. There exists a map $c \to p_c$ from L^+ to $C(K)$ such that for every $c \in L^+$, $q_c(lk) = c(k^{-1}lk)p_c(k)$ is a positive eigenfunction for Σ with $q_c(e) = 1$. Moreover, if f is a (point) divisible function there exists a Borel measure μ on L^+ such that for all $x \in G$,*

$$f(x) = \int_{L^+} q_c(x)\mu(dc). \tag{4.10}$$

REMARK 4.11. If $K = \{e\}$, that is, if $G = L$ is abelian, then (4.10) is a *bilateral Laplace transform* on G, and essentially the same theorem was obtained earlier in [19].

REMARK 4.12. If every $\phi \in \Sigma$ has the property that $\phi(kx) = \phi(x)$, $k \in K$, $x \in G$, then every $f \in \mathcal{S}$ obviously has the same property. This implies $p_c \equiv 1$ for every $c \in L^+$, and (4.10) becomes

$$\begin{aligned} f(x) = f(lk) &= f(k^{-1}lk) \\ &= \int_{L^+} c(k^{-1}lk)\mu(dc) \\ &= \int_{L^+} c(l)\mu_k(dc) \end{aligned} \tag{4.13}$$

where μ_k is the image of μ under the automorphism of L^+ adjoint to $l \to k^{-1}lk$ on L.

For example, if G is the group of isometries of $\mathbb{R}^n$, then L is isomorphic to $\mathbb{R}^n$. If a is a linear isometry of $\mathbb{R}^n$ (an element of K), the adjoint automorphism of L^+ is $w \to aw$ (L^+ is identified with $\mathbb{R}^n$). Thus, (4.13) is

$$f(lk) = \int_{\mathbb{R}^n} \exp\langle l, kw\rangle \mu(dw) \tag{4.14}$$

in which L and $\mathbb{R}^n$ are identified.

If we allow Σ to operate on the right ($f \to f * \phi$), and if we define $\Sigma^* = \{\phi^* \mid \phi^*(x) = \phi(x^{-1}), \text{ some } \phi \in \Sigma\}$, then f is divisible for the right action of Σ if and only if f^* is divisible for the left action of Σ^*. If, for example, $\phi(xk) = \phi(x)$, $k \in K$, then (4.14) applies to f^* and Σ^*. If $x = lk$, then

$$x^{-1} = k^{-1}l^{-1} = k^{-1}l^{-1}kk^{-1},$$

and (4.14) becomes

$$\begin{aligned} f(lk) &= f^*(k^{-1}l^{-1}kk^{-1}) \\ &= \int_{\mathbb{R}^n} \exp\langle k^{-1}l^{-1}k, k^{-1}w\rangle \mu(dw) \\ &= \int_{\mathbb{R}^n} \exp\langle l^{-1}, w\rangle \mu(dw) \end{aligned} \tag{4.15}$$

and f is a bilateral Laplace transform on $\mathbb{R}^n$ lifted to G.

Finally, if $\Sigma = \Sigma^*$ (for example, if Σ is the set of all $\phi \in \mathscr{K}$ such that $\phi(kx) = \phi(x) = \phi(xk)$, $k \in K$, which is abelian), then both (4.14) and (4.15) apply. It follows μ is a radial measure and $f(l) = f(l^{-1})$. If $J_t(r)$ denotes the Bessel function of order t, and if we regard f as a function of $x \in \mathbb{R}^n$, ($f(x) = f(kx)$ all k) then there exists a measure λ on $[0, \infty)$ such that $f(x) = F(|x|)$, where

$$F(r) = r^{-\frac{1}{2}(n-2)} \int_0^\infty J_{\frac{1}{2}(n-2)}(irs)\lambda(ds). \tag{4.16}$$

See [16, p. 155]. (Without the i this transform is called the Hankel transform.) Thus functions which are *left- and right-divisible on* G are characterized by the transform (4.16).

We turn now to the case of a connected noncompact semi-simple Lie group with finite center. In this setting the hard work of identifying the eigenfunctions has already been done in [6] (on which the preceding discussion is based). In what follows we shall use results of Furstenberg [6] and Karpelevič [11] to held to describe the representation for $\mathscr{S}$.

Fix an Iwasawa decomposition, $G = KAN$, for G. K is a maximal compact subgroup (G is assumed to have finite center), A is a vector group, N is a simply connected nilpotent group, and the map $(k, a, n) \to kan$ is a diffeomorphism from $K \times A \times N$ onto G. The Lie algebras of G, K, A, and N will be denoted by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. The space of real-valued linear functionals on $\mathfrak{a}$ is denoted by $\mathfrak{a}^*$. If $\lambda \in \mathfrak{a}^*$, define $\mathfrak{g}^\lambda \subseteq \mathfrak{g}$ by $\mathfrak{g}^\lambda = \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X,\ H \in \mathfrak{a}\}$. If $\mathfrak{g}^\lambda \neq \{0\}$, λ is a *root* of the pair $(\mathfrak{g}, \mathfrak{a})$. If Λ is the set of roots, then $\mathfrak{g}$ is a direct sum, $\mathfrak{g} = \sum_{\lambda \in \Lambda} \mathfrak{g}^\lambda$. Moreover, there is a linear ordering of $\mathfrak{a}^*$ such that if $\Lambda^+ = \{\lambda \in \Lambda \mid \lambda > 0\}$ with respect to this ordering, then $\mathfrak{n} = \sum_{\lambda \in \Lambda^+} \mathfrak{g}^\lambda$. Define $\rho \in \mathfrak{a}^*$ by $\rho = 1/2 \sum_{\lambda \in \Lambda^+} (\dim \mathfrak{g}^\lambda)\lambda$.

By the Iwasawa decomposition, each $g \in G$ has a unique expression as $g = k(g)a(g)n(g)$ with $k(g) \in K$, $a(g) \in A$, and $n(g) \in N$. The exponential map $\exp\colon g \to G$ is a diffeomorphism of $\mathfrak{a}$ onto A whose inverse will be denoted by log. Define $H\colon G \to \mathfrak{a}$ by $H(g) = \log a(g)$. Then for each $\nu \in \mathfrak{a}^*$ define

$$f_\nu(g) = \exp - (\rho + \nu)(H(g)). \tag{4.17}$$

Let M be the centralizer of A in K, the closed subgroup of K consisting of the elements which commute with each element of A. $S = AN$ is a (closed) solvable subgroup of G with commutator N, and M has an alternate description as the normalizer of S in K ($= \{k \in K \mid kS = Sk\}$). Finally, $P = MAN$ is also a closed subgroup of G, and N (as well as S) is normal in P because N is the commutator of S. Therefore, if $g \in G$ and $m \in M$ we have

$$gm = k(g)a(g)n(g)m = k(g)ma(g)m^{-1}n(g)m = ka(g)n,$$

$k = k(g)m$, $n = m^{-1}n(g)m \in N$. Thus $H(gm) = H(g)$. Let X be the homogeneous space $X = G/P$. X is compact because K is transitive on X, (since $KP = G$). We define a function $\sigma_\nu(\cdot\,,\cdot)$ on $G \times X$ by

$$\sigma_\nu(g, kP) = f_\nu(gk). \tag{4.18}$$

Since $kP = k'P$ if and only if $k' = km$ for some $m \in M$, the fact σ_ν is well defined follows from the discussion above.

X is called the *Furstenberg boundary* of G ([5], [13]). X is unique in the sense that if P' is associated to a second Iwasawa decomposition for G as P was to our given decomposition, then P' is conjugate to P in G.

The functions (4.18) are called *K-multipliers* by Furstenberg because they satisfy the relations

4.19) $$\sigma_\nu(gg',x) = \sigma_\nu(g,g'x)\sigma_\nu(g',x) \qquad (g,g' \in G, x \in X)$$

and

(4.20) $$\sigma_\nu(k,\,\cdot\,) \equiv 1 \qquad (k \in K)$$

(see [6]). It is easy to see that every positive function satisfying (4.19)–(4.20) has the form (4.18).

We return now to the study of convolution operators on G. If $\phi \in \mathscr{K}$, $\nu \in \mathfrak{a}^*$, $x_0 \in X$, there exists a unique $q_\nu \in C(X)$ such that $q_\nu(x_0) = 1$, $q_\nu > 0$ on X, and for each $x \in X$, $F_\nu(\,\cdot\,, x)$ is an eigenfunction for ϕ where

(4.21) $$F_\nu(g,x) = \sigma_\nu(g,x)\,\frac{q_\nu(gx)}{q_\nu(x)}$$

(see [6]). It is easy to proceed from this statement to a similar statement for arbitrary abelian sub-semigroups $\Sigma \subseteq \mathscr{K}$.

LEMMA 4.22. *Let Σ be an abelian sub-semigroup of $\mathscr{K}$. For each $\nu \in \mathfrak{a}^*$ there exists a unique $q_\nu \in C(X)$, $q_\nu > 0$, $q_\nu(x_0) = 1$ such that the function F_ν in* (4.21) *is an eigenfunction for Σ.*

PROOF. Of course since q_ν is already unique for a cyclic semigroup Σ, it has only to be proved that q_ν exists. To this end fix any $\psi \in \mathscr{K}$ and $q \in C(X)$, $q > 0$, and let F_ν be as in (4.21). We compute

$$\begin{aligned} \psi * F_\nu(g,x) &= \int_G \psi(gh^{-1})F_\nu(h,x)dh \\ &= \int_G \psi(h^{-1})F_\nu(hg,x)dh \\ &= \int_G \psi(h^{-1})F_\nu(h,gx)F_\nu(g,x)dh \\ &= \alpha(gx)F_\nu(g,x) \end{aligned}$$

where

$$\alpha(gx) = \int_G \psi(h^{-1})F_\nu(h,gx)dh.$$

Notice that $\alpha > 0$ on X. We can therefore write

(4.23) $$\psi * F_\nu(g,x) = \alpha(x)\left\{\frac{\alpha(gx)}{\alpha(x)}F_\nu(g,x)\right\}.$$

Now if F_ν is an eigenfunction for some $\phi \in \mathscr{K}$, and if $\psi * \phi = \phi * \psi$, then $\psi * F_\nu$ is also an eigenfunction for ϕ. By the uniqueness of q_ν in (4.21) (for ϕ), it must be by (4.23) that $\alpha(\cdot)$ is constant. Therefore $\psi * F_\nu = \alpha F_\nu$, and F_ν is an eigenfunction for ψ. The lemma follows because q_ν exists for any given $\phi \in \Sigma$ by Furstenberg's result cited above.

Furstenberg's main result in [6] is that every extremal eigenfunction (for cyclic Σ) has the form (4.21) for some $\nu \in \mathfrak{a}^*$. Therefore, by the results of Section 3, every $f \in \mathscr{S}$ has a representation

$$f(g) = \int_{\mathfrak{a}^* \times X} \sigma_\nu(g,x)\frac{q_\nu(gx)}{q_\nu(x)}\mu(d(\nu,x)). \tag{4.24}$$

We will eventually apply a result of Karpelevič to show there is a closed subset $\mathscr{C} \subseteq \mathfrak{a}^*$, independent of Σ, such that μ can be taken to be concentrated on $\mathscr{C} \times X$. Later (in Section 8), we will see that this μ is unique.

Let M' be the normalizer of A in $\mathscr{K}$. $W = M'/M$ is the *Weyl group*, a finite group which acts on A (and $\mathfrak{a}$) by $w(a) = \operatorname{ad} m_w a$, $a \in A$, $w \in W$, $w \sim m_w M$. We shall also write $w(\nu)$, $\nu \in \mathfrak{a}^*$, for the transpose of w.

Furstenberg's identification of the extremal eigenfunctions for cyclic semigroups together with the fact that Σ has the same eigenfunctions as any of its cyclic sub-semigroups tells us that every extremal of $\mathscr{S}$ is a multiple of F_λ for some $\lambda \in \mathfrak{a}^*$. In order to further identify the extremals let $\nu \in \mathfrak{a}^*$ be fixed, and let there be a representation

$$F_\nu(g,x) = \int_{*\times X} F_\lambda(g,y)\mu(d(\lambda,y)) \tag{4.25}$$

with μ a Borel probability measure on $\mathfrak{a}^* \times X$. Define $\beta(\lambda)$ to be the integral of q_λ over X with respect to the unique K-invariant Borel probability measure on X. If dk is normalized Haar measure on K, then $\beta(\lambda) = \int_K q_\lambda(kx)dk$ for any $x \in X$. If we now replace g by kg in (4.25) ($k \in K$), and then integrate with respect to dk, then because $\sigma_\lambda(kg,z) = \sigma_\lambda(g,z)$, we obtain a representation

$$\sigma_\nu(g,x) = \int_{\mathfrak{a}^* \times X} \sigma_\lambda(g,y)\mu_0(d(\lambda,y)) \tag{4.26}$$

in which μ_0 is the measure

$$\mu_0(d(\lambda,y)) = \frac{\beta(\lambda)}{\beta(\nu)}\,\frac{q_\nu(x)}{q_\lambda(y)}\mu(d(\lambda,y)).$$

Notice that μ_0 is a point mass if and only if μ is a point mass. Therefore, if ν is such that σ_ν is an extremal K multiplier, then $\sigma_\lambda(\cdot, y) \equiv \sigma_\nu(\cdot, x)$ a.e. μ_0. According to Karpelevič [11, Sect. 17], $\sigma_\lambda(\cdot, y) \equiv \sigma_\nu(\cdot, x)$ if and only if $\lambda = \nu$ and $y = x$. To investigate the nonextremals it is necessary to have the following lemma whose proof is deferred until the end of this section.

LEMMA 4.27. *If $\nu \in \mathfrak{a}^*$, and if μ_0 is a measure on $\mathfrak{a}^* \times X$ such that* (4.26) *holds, then μ_0 is concentrated on $W\nu \times X$, where $W\nu$ is the orbit of ν under W.*

If σ_λ is a K-multiplier, define the *spherical function* ϕ_λ by

$$\phi_\lambda(g) = \int_K \sigma_\lambda(g, kx)dk.$$

A theorem of Harish-Chandra ([8, Chap. X]) asserts that $\phi_\lambda = \phi_{\lambda'}$ if and only if $\lambda' \in W\lambda$, and therefore the partition of $\mathfrak{a}^*$ induced by $\lambda \to \phi_\lambda$ is the partition of $\mathfrak{a}^*$ into the orbits of W. Fixing a positive spherical function ϕ, Furstenberg [6] proves there is a unique K-multiplier σ such that

$$\phi(g) = \int_K \sigma(g, kx)dk$$

and if ψ is any positive solution to

$$\int \psi(gkg')dk = \phi(g)\psi(g')$$

then ψ admits a representation

$$\psi(g) = \int_X \sigma(g, x)m(dx) \tag{4.28}$$

with m *a* (unique) Borel measure on X. Let λ be such that $\sigma = \sigma_\lambda$ in (4.28). Then by the discussion preceding the statement of Lemma 4.27, $F_\lambda(\cdot, x)$ is an extremal of $\mathscr{S}$ for every $x \in X$. We will now prove that if $\lambda' \in W\lambda$, $\lambda' \neq \lambda$, then for every x, $F_{\lambda'}(\cdot, x)$ is *not* an extremal of $\mathscr{S}$.

LEMMA 4.29. *Each orbit of W in $\mathfrak{a}^*$ contains a unique element λ_0 (namely the one described above) such that for each $y \in X$, $F_{\lambda_0}(\cdot, y)$ is an extremal of $\mathscr{S}$. λ_0 is independent of Σ.*

PROOF. Fix an orbit of W in $\mathfrak{a}^*$, and let λ_0 be the unique element in the orbit such that $\sigma = \sigma_{\lambda_0}$ in (4.28). Given $\lambda \in W\lambda_0$ let m be the measure on X in (4.28) for the choice $\psi(g) = \sigma_\lambda(g, x_0)$, x_0 the identity coset of P in X. $\sigma_\lambda(\cdot, x_0)|_P$ is a character, and therefore for any $t \in P$ if we replace g by gt in (4.28) and simplify, we obtain a representation

$$\sigma_\lambda(g, x_0) = \int_X \frac{\sigma_{\lambda_0}(gt, x)}{\sigma_\lambda(t, x_0)} m(dx) \tag{4.30}$$

$$= \int_X \sigma_{\lambda_0}(g, tx) \frac{\sigma_{\lambda_0}(t, x)}{\sigma_\lambda(t, x_0)} m(dx).$$

Define $m_t(\cdot)$ by $m_t(E) = m(t^{-1}E)$, $E \subseteq X$ Borel. Rewriting (4.30) in terms of m_t we have

$$\sigma_\lambda(g, x_0) = \int_X \sigma_{\lambda_0}(g, y) \frac{\sigma_{\lambda_0}(t, t^{-1}y)}{\sigma_\lambda(t, x_0)} m_t(dy)$$

$$= \int_X \sigma_{\lambda_0}(g, y) \frac{1}{\sigma_\lambda(t, x_0)\sigma_{\lambda_0}(t^{-1}, y)} m_t(dy).$$

Since the representation (4.28) is unique, it must be that

$$m_t(dy) = \sigma_\lambda(t, x_0)\sigma_{\lambda_0}(t^{-1}, y)m(dy).$$

Given a positive $q \in C(X)$, define $r(g)$, $g \in G$, by

$$r(g)\sigma_\lambda(g, x_0) = \int_X \sigma_{\lambda_0}(g, y)q(gy)m(dy).$$

Given $t \in P$ we use the computation for m_t to find

$$\sigma_\lambda(gt, x_0)r(gt) = \int_X \sigma_{\lambda_0}(gt, y)q(gty)m(dy)$$

$$= \int_X \sigma_{\lambda_0}(g, ty)\sigma_{\lambda_0}(t, y)q(gty)m(dy)$$

$$= \int_X \sigma_{\lambda_0}(g, y)\sigma_{\lambda_0}(t, t^{-1}y)q(gy)m_t(dy)$$

$$= \int_X \sigma_{\lambda_0}(g, y)\sigma_{\lambda_0}(t^{-1}, y)^{-1}q(gy)\sigma_\lambda(t, x_0)\, \sigma_{\lambda_0}(t^{-1}, y)m(dy)$$

$$= \sigma_\lambda(t, x_0)\int_X \sigma_{\lambda_0}(g, x)q(gy)m(dy)$$

$$= \sigma_\lambda(t, x_0)\sigma_\lambda(g, x_0)r(g)$$

$$= \sigma_\lambda(gt, x_0)r(g).$$

Therefore $r(g) = r(gt)$, and r can be regarded as a continuous function on X. If $q = q_{\lambda_0}$, then because Σ operates on the left, $\sigma_\lambda(\cdot, x_0)r(\cdot)$ is an eigenfunction for Σ. The space of eigenfunctions belonging to a fixed character on Σ is invariant under right translations, and therefore since $\sigma_\lambda(k, \cdot) \equiv 1$, $k \in K$, we have that

$F_\lambda^0(g, x) = \sigma_\lambda(g, x)\, r(gx)/r(x)$ is an eigenfunction for Σ. By the uniqueness statement in Lemma 4.22, $F_\lambda{}^0 = F_\lambda$. Therefore we have a representation

$$F_\lambda(g, x_0) = \int_X F_{\lambda_0}(g, y) q_{\lambda_0}(y) m(dy).$$

Now $F_\lambda(\cdot\,, x)$ is an extremal of $\mathscr{S}$ for one x if and only if $F_\lambda(\cdot\,, x)$ is an extremal for all x (right translation preserves the property of being an extremal). Therefore, if $F_\lambda(\cdot\,, x)$ is an extremal for some x, $F_\lambda(\cdot\,, x_0)$ is an extremal and $q_{\lambda_0}(y)m(dy)$ must be a point mass at some $x_1 \in X$. In particular, $m(\cdot)$ is a point mass at x_1, and we have that $\sigma_\lambda(\cdot\,, x_0) = \sigma_{\lambda_0}(\cdot\,, x_1)$. Therefore, by the Karpelevič result mentioned before Lemma 4.27, $\lambda = \lambda_0$ (and $x_0 = x_1$), and the lemma is proved.

We now recall the identification by Karpelevič of the distinguished elements of the orbits of W. Let $B(\cdot\,,\cdot)$ be the Killing form for G. B is positive definite on $\mathfrak{a}$. Given $\lambda \in \mathfrak{a}^*$, there exists a unique $Q_\lambda \in \mathfrak{a}$ such that $\lambda(H) = B(H, Q_\lambda)$, $H \in \mathfrak{a}$. Recall that Λ^+ is the set of positive roots with respect to the fixed ordering of $\mathfrak{a}^*$, and define $\mathscr{C}_0 \subseteq \mathfrak{a}^*$ by

$$\mathscr{C}_0 = \{\lambda' \in \mathfrak{a}^* \mid \lambda'(Q_\lambda) > 0,\ \lambda \in \Lambda^+\}.$$

Finally, define $\mathscr{C}$ to be the closure of $\mathscr{C}_0$ in $\mathfrak{a}^*$. Then by [11, Th. 17.2.1], if $\lambda \in \mathfrak{a}^*$ and if λ_0 is the (unique) element of $W\lambda$ which belongs to $\mathscr{C}$, there exists for each $x \in X$ a measure $m = m_{x,\lambda}$ on X such that (4.28) holds with $\sigma = \sigma_{\lambda_0}$ and $\psi(\cdot) = \sigma_\lambda(\cdot\,, x)$. (The measure m is actually computed explicitly if $\lambda_0 \in \mathscr{C}_0$. See, for example, the exposition in [22, Sect. 9.1.6, particularly the remark following Prop. 9.1.6.6].)

THEOREM 4.31. *With notation and assumptions as above, there exists for each $f \in \mathscr{S} = \mathscr{S}(\Sigma)$ a measure μ on $\mathscr{C} \times X$ such that for all $g \in G$*

$$f(g) = \int_{\mathscr{C}\times X} \sigma_\lambda(g, x)\, \frac{q_\lambda(gx)}{q_\lambda(x)}\, \mu(d(\lambda, x)). \tag{4.32}$$

For each $\lambda \in \mathfrak{a}^*$, q_λ is determined as the unique positive element of $C(X)$ such that $q_\lambda(x_0) = 1$ (x_0 fixed in X) and $\psi(g) = \sigma_\lambda(g, x) q_\lambda(gx)$ is an eigenfunction of Σ for every $x \in X$.

REMARK 4.33. As in Remark 4.12, if some $\phi \in \Sigma$ is left-invariant under K, then every $f \in \mathscr{S}$ has the same property. Since $\sigma_\lambda(kg, x) = \sigma_\lambda(g, x)$, $k \in K$, $g \in G$ $x \in X$ it follows $q_\lambda(kx) = q_\lambda(x)$. Since K is transitive on X, $q_\lambda \equiv 1$. Thus (4.32) becomes

(4.34) $$f(g) = \int_{\mathscr{C} \times X} \sigma_\lambda(g, x)\mu(d(\lambda, x)).$$

If $\Sigma = \Sigma^*$ (see Remark 4.12), and if $\mathscr{S}_0$ is the space of *left- and right-divisible* functions, then each $f \in \mathscr{S}_0$ has a representation

$$f(g) = \int_{\mathscr{C}} \phi_\lambda(g)\mu_0(d\lambda)$$

where we recall that $\phi(\cdot) = \int_K \sigma_\lambda(\cdot, kx)dk$.

EXAMPLE 4.35. $G = SL(n, \mathbb{R})$, $n \geqq 2$. Here the Gram-Schmidt process, applied to the columns of $g \in G$, leads to the Iwasawa decomposition $G = KAN$ in which $K = SO(n, \mathbb{R})$, A is the group of diagonal matrices with positive diagonal entries and determinant 1, and N is the group of upper triangular matrices with 1s on the diagonal.

The Lie algebra $\mathfrak{a}$ can be identified as the set of diagonal matrices a, given in terms of the diagonal entries $a = (a_1, a_2, \cdots, a_n)$, and having trace $a = a_1 + \cdots + a_n = 0$. Define $\lambda_i \in \mathfrak{a}^*$ by $\lambda_i(a) = a_i$. In terms of the natural lexicographical ordering of $\mathfrak{a}^*$, the set of positive roots of $\mathfrak{a}^*$ is $\Lambda^+ = \{\lambda_i - \lambda_j \mid 1 \leqq i < j \leqq n\}$. Each $\lambda \in \mathfrak{a}^*$ is expressible as $\lambda = x_1\lambda_1 + \cdots + x_n\lambda_n$ with $x_1 + \cdots + x_n = 0$ because $\lambda_1 + \cdots + \lambda_n \equiv 0$. Therefore $\mathfrak{a}$ and $\mathfrak{a}^*$ can be identified in terms of the Killing form for g, $B(a_1, a_2) = \text{trace}\, a_1 a_2^t$, $a^t =$ transpose of a. Thus Q_λ, $\lambda = \lambda_i - \lambda_j \in \Lambda^+$, is the matrix $Q_\lambda = (0, \cdots, 0, 1, 0, \cdots, -1, 0, \cdots, 0)$ where the 1 occurs in the ith place, the -1 in the jth place. Thus, $\mathscr{C}$ is the set $\mathscr{C} = \{Q = (x_1, \cdots, x_n) \mid x_1 \geqq x_2 \geqq \cdots \geqq x_n\}$ because $(Q, Q_\lambda) = x_i - x_j$ must be $\geqq 0$ for $i < j$.

In order to compute the multipliers σ_λ, $\lambda \in \mathfrak{a}^*$, it is necessary to make a remark on the Iwasawa decomposition. If g is an $n \times n$ matrix, $C_i(g)$ will denote the ith column of g as a vector in $\mathbb{R}^n$. It is easy to check (by consideration of the Gram-Schmidt process, for example) that the ith component of $a(g)$ (recall $g = k(g)a(g)n(g)$) is the euclidean distance from $C_i(g)$ to 0 if $i = 1$ and to the span of $C_1(g), \cdots, C_{i-1}(g)$ if i > 1. In terms of the exterior product on $\mathbb{R}^n$ this distance is

(4.36) $$\delta_i(g) = \frac{\| C_1(g) \wedge \cdots \wedge C_i(g) \|}{\| C_1(g) \wedge \cdots \wedge C_{i-1}(g) \|}$$

where if $i = 1$ the denominator is interpreted as 1. Thus, if $g \in G$,

$$H(g) = (\log \delta_1(g), \cdots, \log \delta_n(g)).$$

Now suppose $\lambda = x_1\lambda_1 + \cdots + x_n\lambda_n$, $\Sigma x_n = 0$, and note

$$\rho = \frac{1}{2} \sum_{1 \leqq i < j \leqq n} (\lambda_i - \lambda_j) = \sum_{i=1}^{n} \frac{n-2i+1}{2}\lambda_i.$$

If $x \in X$ is $x = kx_0$, $k \in I$, then

$$\begin{aligned}\sigma_\lambda(g,x) &= \exp - (\lambda + \rho)(H(gk)) \\ &= \prod_{i=1}^{n} \delta_i(gk)^{-(x_i + \frac{1}{2}(n-2i+1))} \\ &= \prod_{i=1}^{n-1} \| C_1(gk) \wedge \cdots \wedge C_i(gk) \|^{x_{i+1} - x_i - 1}.\end{aligned}$$

(Of course, $\| C_1(gk) \wedge \cdots \wedge C_n(gk) \| = \det gk = 1$.) Assume now that some $\phi \in \Sigma$ is left-invariant under K. The representation (4.34) becomes

$$f(g) = \int_{(\mathbb{R}^+)^{n-1} \times X} \prod_{i=1}^{n-1} \| C_1(gk) \wedge \cdots \wedge C_i(gk) \|^{-1-\alpha_i} \mu(d(\alpha,x))$$

where we have identified $\mathscr{C}$ and $(\mathbb{R}^+)^{n-1}$ via the map

$$x_1\lambda_1 + \cdots + x_n\lambda_n \to (x_1 - x_2, \cdots, x_{n-2} - x_{n-1}) = (\alpha_1, \cdots, \alpha_{n-1}).$$

Consider the case $n = 2$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (v_1, v_2)$, where $v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$, $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$, and if $k = k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $0 \leqq \theta < 2\pi$, one computes

$$\| C_1(gk) \| = \begin{cases} |\sin\theta| \, \| \cot\theta \, v_1 - v_2 \| & \theta \neq 0, \pi \\ \| v_1 \| & \theta = 0, \pi \end{cases}$$

and in any case $\| C_1(gk(\theta + \pi)) \| = \| C_1(gk(\theta)) \|$. Set $t = \cot\theta$ and obtain the representation

$$f(g) = \int_{t=-\infty}^{\infty} \int_{\alpha=0}^{\infty} \left\{ \frac{1+t^2}{\| tv_1 - v_2 \|} \right\}^{\frac{1}{2}(1+\alpha)} \mu(d(\alpha,t)) + \int_0^{\infty} \| v_1 \|^{-1-\alpha} \nu(d\alpha). \tag{4.37}$$

The measure on $(-\infty, \infty)$ which corresponds to $d\theta/2\pi = dz/2\pi iz$ on the unit circle under the fractional linear transformation $z \to i(1 + z/1 - z)$ of the unit disc onto the upper half plane $\mathscr{H}$ is $dt/\pi(1+t^2)$. The Poisson kernel for $\mathscr{H}$ with respect to this measure is

$$\begin{aligned} P(x+iy,t) &= \frac{y(1+t^2)}{(x-t)^2+y^2} \\ &= \frac{(1+t^2)}{\frac{1}{y}(x-t)^2+y} \\ &= \frac{1+t^2}{\frac{x^2}{y}-2t\frac{x}{y}+\frac{t^2}{y}+y}. \end{aligned}$$

If $g \in G$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and as a fractional linear transformation acting on $i \in \mathscr{H}$,

$$\begin{aligned} g^{-1}i &= \frac{di-b}{-ci+a} \\ &= \frac{1}{\|v_1\|^2}i - \frac{(v_1,v_2)}{\|v_1\|^2}. \end{aligned}$$

Then, writing $g^{-1}i = x+iy$, we have by a simple computation

$$\begin{aligned} \frac{x^2}{y}+y &= \frac{1+(v_1,v_2)^2}{\|v_1\|^2} \\ &= \|v_2\|^2 \\ -2\frac{x}{y} &= 2(v_1,v_2) \\ \frac{1}{y} &= \|v_1\|^2 \end{aligned}$$

and therefore

$$P(g^{-1}i,t) = \frac{(1+t^2)}{\|v_2\|^2+2t(v_1,v_2)+t^2\|v_1\|^2}. \tag{4.38}$$

Also, define $P(g^{-1}i,\infty) = \|v_1\|^{-2}$. Using (4.38) in (4.37) we find

$$f(g) = \int_{t=-\infty}^{t=\infty}\int_{\alpha=0}^{\infty} P(g^{-1}i,t)^{\frac{1}{2}(1+\alpha)}\,\mu(d(\alpha,t)) + \int_0^{\infty} P(g^{-1}i,\infty)^{\frac{1}{2}(1+\alpha)}\nu(d\alpha) \tag{4.37'}$$

is the most general element of $\mathscr{S}$. In particular we see that the extremal K multipliers for $SL(2,\mathbb{R})$ have the form $P(g^{-1}i,t)^{1/2+\alpha/2}$ for some $t\in\{\infty\}\cap(-\infty,\infty)$ and $\alpha\geqq 0$ as was proved in [6, Sect. 7].

We conclude with the proof of Lemma 4.27. (For a less elementary proof see [11, p. 190].)

Let ϕ be any non-negative, compactly supported, continuous function on G such that $\phi(kg)=\phi(g)$, $k\in K$. If $\lambda\in\mathfrak{a}^*$, $x\in X$, then

$$\begin{aligned}\phi*\sigma_\lambda(g,x) &= \int \phi(gh^{-1})\sigma_\lambda(h,x)dh\\ &= \int \phi(h^{-1})\sigma_\lambda(h,gx)dh\sigma_\lambda(g,x)\end{aligned}$$

by the multiplier equation. Replace g by kg, use the fact $\phi(kg)=\phi(g)$, and integrate with respect to dk. We obtain that $\phi*\sigma_\lambda(g,x)=\alpha_\lambda\sigma_\lambda(g,x)$ where $\alpha_\lambda=\alpha_\lambda(\phi)$ is

$$\alpha_\lambda=\int_G \phi(h^{-1})\phi_\lambda(h)dh$$

with ϕ_λ the spherical function corresponding to σ_λ. From Fubini's theorem and (4.26) we obtain (setting $g=1$)

$$\alpha_\nu=\int_{\mathfrak{a}^*\times X}\alpha_\lambda\mu_0(d(\lambda,y))$$

and applying the same argument to $\phi^{(n)}$, the n-fold convolution of ϕ,

$$\alpha_\nu^n=\int_{\mathfrak{a}^*\times X}\alpha_\lambda^n\mu_0(d(\lambda,y)).$$

Since a geometric sequence is uniquely represented as a Stieltjes moment sequence $\alpha_\nu(\phi)=\alpha_\lambda(\phi)$ a.e. $\mu_0(d(\lambda,y))$. Since ϕ is arbitrary, it follows that $\phi_\nu=\phi_\lambda$ a.e. μ_0 In other words, by the Harish-Chandra result mentioned earlier, μ_0 is supported on $W\nu\times X$. The lemma is proved.

5. Application to potential theory

Let Ω be a bounded region in $\mathbb{R}^n$, $n\geqq 1$, and let $d(x,\Omega^c)$, $x\in\Omega$, denote the distance from x to the boundary of Ω. Given a function $\delta(\cdot)$ on Ω, $0<\delta(x)\leqq d(x,\Omega^c)$, set up a kernel $P=P_\delta$ on $\Omega\times\Omega$ defining

$$P_\delta(x,y)=\chi_{B(x)}(y)\,|B(x)|^{-1}$$

where $|\cdot|$ denotes volume and

$$B(x) = \{y \mid \| y - x \| < \delta(x)\}.$$

Let there be given a Lipschitz function r on Ω with Lipschitz constant 1, such that $0 < r(x) \leqq d(x, \Omega^c)$. Then assume δ is a Borel function which for some $\alpha > 0$ satisfies $\alpha r(x) \leqq \delta(x) \leqq (1 - \alpha) r(x)$. (This condition on δ is used also in [18].)

LEMMA. 5.1. *With notation and assumptions as above, we have* $P^{(2)} = P \circ P \geqq \beta P$, *where* $\beta = \beta(n, \alpha) > 0$.

PROOF. If $x \in \Omega$, and if $y \in B(x)$, then $|r(x) - r(y)| \leqq \| x - y \| < \delta(x)$. From this we obtain $r(x) - \delta(x) < r(y) < r(x) + \delta(x)$, or $(\alpha/(1 - \alpha))\delta(x) < r(y) < ((\alpha + 1)/\alpha)\, \delta(x)$. Therefore

$$\frac{\alpha^2}{1 - \alpha}\delta(x) < \delta(y) < \frac{1 - \alpha^2}{\alpha}\delta(x).$$

Now if $y \in B(x)$, and if $z \in B(x)$ is such that $\| z - y \| < (\alpha^2/(1 - \alpha))\delta(x)$, then $z \in B(y)$ and $P(y, z) \geqq [\alpha/(1 - \alpha^2)]^n P(x, z)$. For a fixed $z \in B(x)$ the set of such y contains a ball of radius $1/2(\alpha^2/(1 - \alpha))\delta(x)$ and therefore

$$P^{(2)}(x, z) \geqq 2^{-n}\left[\frac{\alpha^2}{1 - \alpha}\right]^n \left[\frac{\alpha}{1 - \alpha^2}\right]^n P(x, z)$$

and the lemma is proved.

The lemma implies Σ is strongly admissible. If $f \in \mathscr{S}$ then because $\{z \mid P^{(n)}(x, z) > 0\} \nearrow \Omega$ for every x, as the reader may easily check, we have for any compact set $B \subseteq \Omega$, $|B| > 0$, that $\sup_n (P^{(n)}f, \chi_B) > 0$ for every $f \in S$. By Theorem 3.15 there exists for every divisible point function f a representation

$$f(x) = \int_E e(x)\mu(de) \tag{5.2}$$

in which $Pe(x) = \lambda_e e(x)$, all e and x, and $(e, \chi_B) = 1$, all e. Fixing $x_0 \in \Omega$, the latter condition can be replaced by $(e, P(x_0, \cdot\,)) = \lambda_e$, and thus E becomes $E = \{e \mid e(x_0) = 1\}$.

REMARK 5.3. Define $\mathscr{S}_\infty = \{f \in \mathscr{S} \mid \sup_n P^{(n)} f(x) < \infty\}$ for some $x \in \Omega$. From (5.2) it follows that $f \in \mathscr{S}_\infty$ if and only if μ is concentrated on the set $\{e \mid \lambda_e \leqq 1\}$. In Section 6 it will be proved μ is unique if $f \in \mathscr{S}_\infty$, and this fact will be assumed in what follows.

REMARK 5.4. Assume Ω is a bounded Lipschitz domain. (Locally $\partial\Omega$ is the graph of a Lipschitz function in some orthogonal coordinate system.) If δ is as restricted above, then by [18, Th. 1.3], any non-negative solution to $Pf = f$ is harmonic. (It is not necessary here to assume δ is measurable, only that it be comparable to r as above. [20, Th. 1.2] asserts for any δ which is locally bounded away from 0 on Ω that if $Pf = f$, and if f is dominated by a positive harmonic function on Ω, then f is harmonic.)

THEOREM 5.5. *Let δ be as restricted above. Let $g \geqq 0$ be such that $Pg \leqq g$. If $f \in \mathscr{S}$ has the property that for each n there exists f_n, $0 \leqq f_n \leqq g$, such that $P^{(n)}f_n = f$, then f is harmonic.*

PROOF. The arguments of Section 2 are easily modified to show that for each n there exists $f_n \in \mathscr{S}$ such that $f_n \leqq g$ and $P^{(n)}f_n = f$. Since $Pg \leqq g$, $f_n \in \mathscr{S}_\infty$ for every n. By the uniqueness of representation for $\mathscr{S}_\infty$ (Remark 5.3) the representing measure for f must be concentrated on $\{e \mid \lambda_e \geqq 1\}$ because necessarily

$$f_n = \int \frac{e}{\lambda_e^n} \mu(de).$$

Therefore, $Pf = f$ (and $f_n = f$ for all n). Harmonicity of f follows from [18, Th. 1.3] (see Remark 5.4).

REMARK 5.6. Let f be a positive harmonic function on Ω. We define a Markov transition kernel, P_f, by

$$P_f(x,y) = \frac{1}{f(x)} P(x,y) f(y).$$

Let $\Lambda = \Omega \times \Omega \times \cdots$ have coordinate functions $x_0, x_1, \cdots$, and let $\mathscr{B} = \mathscr{B}(x_0, x_1, \cdots)$ be the σ-field they generate. The *shift operator* $x_n(T\omega) = x_{n+1}(\omega)$ is used to define the *tail σ-field* $\mathscr{B}_\infty = \bigcap_{k=0}^{\infty} T^{-k}\mathscr{B}$ and the *invariant σ-field* $\mathscr{B}_I = \{A \in \mathscr{B} \mid T^{-1}A = A\} \subseteq \mathscr{B}_\infty$. Let μ_x^f, $x \in \Omega$, be the probability measure on $\mathscr{B}$ corresponding to the random walk on Ω starting from x and governed by P_f (see [20]). [20, Th. 2.4] asserts that if $\delta(\cdot)$ is locally bounded away from 0 on Ω, and if Ω is a bounded Lipschitz domain, then if f is an extremal of the cone of positive harmonic functions on Ω, $\mathscr{B}_I$ is μ_x^f trivial (that is, every set has measure 0 or 1). In the same setting, but with $\delta(\cdot)$ subject to the more severe restrictions of Theorem 5.5, it is possible to prove $\mathscr{B}_\infty$ is μ_x^f trivial. Indeed, suppose $A \in \mathscr{B}_\infty$ and let $E(\chi_A \mid x_0, \cdots, x_n) = \alpha_n(x_n)$ be the conditional expectation of χ_A with respect to $x_0, \cdots, x_n$. (Since $A \in \mathscr{B}_\infty$ and x_n is a Markov chain, the conditional expectation depends only upon x_n.) The functions $\alpha_n(\cdot)$ satisfy $P_f\alpha_{n+1} = \alpha_n$ and therefore $P_f^{(n)}\alpha_n = \alpha_0(x) = \mu_x^f(A)$. Then also $P^{(n)}\alpha_n f = \alpha_0 f$ and $0 \leqq \alpha_n f \leqq f$. By Theorem

5.5, $\alpha_0 f$ is harmonic and $\alpha_n \equiv \alpha_0$. Since f is extremal and $0 \leqq \alpha_0 f \leqq f$, α_0 is constant on Ω. By the martingale theorem $\alpha(x_n) = E(\chi_A \mid x_0, \cdots, x_n) \to \chi_A$ a.e. μ_x^f, and therefore $\alpha_0 \equiv 0$ or $\alpha_0 \equiv 1$. It follows $\mu_x^f(A) \equiv 0$ (in x) or $\mu_x^f(A) \equiv 1$.

REMARK 5.7. It would be interesting to know when triviality on $\mathscr{B}_I$ implies triviality on $\mathscr{B}_\infty$. It is possible to prove a result slightly more general than (5.6) by establishing for a larger class of δ's that each $\mathscr{B}_\infty$ set differs by a set of μ_x^f measure 0 from a $\mathscr{B}_I$ set ([21]). We omit the details.

REMARK 5.8. ([21].) It is possible to apply the argument of Blackwell-Freedman [2] to prove that when $\mathscr{B}_\infty$ is μ_x^f trivial, then for all $x, x' \in \Omega$

$$\lim_{n\to\infty} \int \left| P_f^{(n)}(x,y) - P^{(n)}(x',y) \right| dy = 0. \tag{5.9}$$

This corresponds to a theorem of Orey for recurrent Markov chains [14].

REMARK 5.10. We do not know if there is any instance in which $\mathscr{S}$ consists entirely of harmonic functions (that is, an instance in which there is only the eigenvalue 1); however there are simple examples to show it cannot always be the case. For example, consider $\Omega = (0,1) \subseteq \mathbb{R}^1$, and define $f(t) = \min(t^{\frac{1}{2}}, (1-t)^{\frac{1}{2}})$, $t \in \Omega$. Fix β, $0 < \beta < 1$, and define $\delta = \delta_\beta$ by $\delta(x) = \min(\beta x, \beta(1-x))$. If $0 < \min(x, 1-x) < 1/4$, we have

$$P_\beta f(x) = \lambda_\beta f(x)$$

where $\lambda_\beta = ((1+\beta)^{3/2} - (1-\beta)^{3/2})/3\beta$. Since f is concave, there is a number $\alpha_\beta < 1$ such that if $1/4 \leqq x \leqq 3/4$, then $P_\beta f(x) \leqq \alpha_\beta f(x)$. Let $\lambda = \max(\lambda_\beta, \alpha_\beta)$. For each x we can decrease $\delta_\beta(x)$ to a value $\delta_0(x)$ such that $P_{\delta_0} f = \lambda f$. Moreover, if $0 < \min(x, 1-x) < 1/4$, δ_0 will clearly have the form $\delta_0(x) = \delta_{\beta_0}(x)$ for some $\beta_0 \leqq \beta$. Therefore there will exist $\alpha > 0$ such that $\alpha \min(x, 1-x) < \delta_0(x) < \delta_0(x) < (1-\alpha)\min(x, 1-x)$.

The following theorem is a *Fatou's theorem* for $\mathscr{S}_\infty$ in the special case $r(x) = d(x, \Omega^c)$ and $\alpha r(x) < \delta(x) < (1-\alpha) r(x)$.

THEOREM 5.12. *Let $\delta(\cdot)$ be a Borel function on the bounded Lipschitz domain Ω such that for some $\alpha > 0$,* $\mathrm{a}d(x, \Omega^c) \geqq \delta(x) \leqq (1-\alpha) d(x, \Omega^c)$, *and let P be the associated operator. If $g \geqq 0$ satisfies $Pg \leqq g$, then $f = Pg$ has non-tangential boundary values* a.e. *on $\partial\Omega$. In particular, if $f \in \mathscr{S}_\infty$, f has non-tangential boundary values* a.e. *on $\partial\Omega$.*

PROOF. $P^{(n)}g$ decreases to a limit which we denote by h, and $Ph = h$. Therefore h is harmonic ([18, Th. 1.3]) and, since the result is known for harmonic h ([9], [10]), we replace g by $g - h$ and assume $h = 0$. Let $\mu_x = \mu_x^1$ $(P_1 = P)$. Then $\lim_{n\to\infty} \int_\Lambda f(x_n(\omega))\mu_x(d\omega) = \lim_{n\to\infty} P^{(n)}f(x) = 0$. Also, because $f \geqq Pf$, $f(x_n(\cdot))$ is a supermartingale [12]. Therefore, $\lim_{n\to\infty} f(x_n(\omega)) = F(\omega)$ exists a.e. μ_x, and

$$\int_\Lambda F(\omega)\mu_x(d\omega) \leqq \lim_{n\to\infty} \int_\Lambda f(x_n(\omega))\mu_x(d\omega) = 0$$

by Fatou's lemma. It follows $\lim_{n\to\infty} f(x_n(\omega)) = 0$ a.e. μ_x.

Fix $x_0 \in \Omega$. For each $Q \in \partial\Omega$ there exists a minimal positive harmonic function f_Q on Ω such that $f_Q(x_0) = 1$ and $f_Q(\cdot)$ vanishes continuously at every $Q' \in \partial\Omega$, $Q' \neq Q$ ([20]). Moreover, there exists a probability measure $m(dQ)$ on $\partial\Omega$ (harmonic measure) such that for all $x \in \Omega$ and $A \in \mathscr{B}$

$$\mu_x(A) = \int_{\partial\Omega} \mu_x^f(A) f_Q(x) m(dQ)$$

([20, (2.1)]). It follows for m-almost all $Q \in \partial\Omega$ that $\lim_{n\to\infty} f(x_n(\omega)) = 0$ a.e. $\mu_x^{f_Q}$. We claim for every such Q that $f(\cdot)$ vanishes nontangentially at Q. For suppose not. Then there exists an $\varepsilon > 0$, a $Q \in \partial\Omega$, and a sequence $\{y_n\} \subseteq \Omega$, $\lim_{n\to\infty} y_n = Q$, such that $f(y_n) \geqq \varepsilon$ and $\| y_n - Q \| \leqq (1/\varepsilon) d(y_n, \Omega^c)$ for all n, and $f(x_n(\omega)) \to 0$ a.e. $\mu_x^{f_Q}$.

The reader can easily modify the proof of Lemma 5.1 to obtain that there exists a number $\beta > 0$ and an integer k such that for all $x, x' \in \Omega$ if $\| x - x' \| < \beta d(x, \Omega^c)$, then $P^{(k+1)}(x', \cdot) \geqq \beta P(x, \cdot)$. (Refer to [18, Lem. 5.1].) This being so, if $f = Pg$ as in the statement of the theorem, and if $\| x - x' \| < \beta d(x, \Omega^c)$ then $f(x') \geqq P^{(k)}f(x') = P^{(k+1)}g(x') \geqq \beta Pg(x) = \beta f(x)$. In particular, if $\| y - y_n \| < \beta d(y_n, \Omega^c)$, then $f(y) \geqq \beta\varepsilon$. Let $U = \cup_n \{y \mid \| y - y_n \| < \beta d(y_n, \Omega^c)\}$. Since $\| y_n - Q \| \leqq (1/\varepsilon) d(y_n, \Omega^c)$, if $B_n = \{y \mid \| y - Q \| \leqq \| y_n - Q \|\}$, there exists a number $\gamma > 0$ such that $| B_n \cap U | \geqq \gamma | B_n |$ ($| \cdot |$ denotes volume). Thus, U has positive upper density at Q in the sense of [20, Sect. 7]. By [20, Th. 7.2], x_n visits U infinitely often for $\mu_x^{f_Q}$ almost all sample paths, and therefore $F \geqq \beta\varepsilon$ a.e. This is a contradiction, and we conclude f has nontangential limit 0 at Q.

The sense of "a.e." in the statement of the theorem is with respect to Hausdorff $n-1$ dimensional measure on $\partial\Omega$. (To the best of our knowledge it is an open question for Lipschitz domains whether mutual absolute continuity of harmonic measure and Lebesue measure prevails.)

6. The Stieltjes moment problem

Let V and S be the sequence spaces of the introduction, and let $\Sigma=\{T^m|m\geqq 1\}$, where $(Tx)_n=x_n+x_{n+1}$. If L is the left shift $(Lx)_n=x_{n+1}$ then $T^m x=(I+L)^m x = \sum_{k=0}^m \binom{m}{k} L^k x$. Letting Ω be the space of non-negative integers and ν be counting measure on Ω, it is obvious from the fact $T^m \geqq I$ for all m that Σ is strongly admissible and support increasing (see Section 3). It is readily checked that every eigenvector for Σ is a multiple of $x_n=\lambda^n$ for some λ, $0<\lambda<\infty$. Applying Theorem 3.14 (see also Remark 2.14) we obtain the following theorem.

THEOREM 6.1. *A sequence $x\in S$ belongs to the cone $T^m S$ for every $m\geqq 1$ if and only if it is the (finite) Stieltjes moment sequence of a finite measure on* $(0,\infty)$.

It follows from Theorem 6.1 that the bounded elements of $\mathscr{S}$ are precisely the moment sequences of measures on $[0,1]$. Of course the latter are also characterized as being the *completely monotone* sequences. (x is completely monotone if $(-\Delta)^m x\in S$ for every $m\geqq 0$, where Δ is the difference operator, $\Delta=L-I$.) Actually, it is easy to verify that every completely monotone sequence belongs to $\mathscr{S}$. For if x is completely monotone, x decreases to a limit c and we may write $x=(x-c)+c$, where c is the constant sequence, and $x-c$ is completely monotone and decreases to 0. Obviously $c\in\mathscr{S}$ and therefore it is to be checked that $x\in\mathscr{S}$ whenever x is completely monotone and decrease to 0. To this end, let y be the sequence $y_n=x_n-x_{n+1}+x_{n+2}-\cdots$, the sum converging for each n Evidently, $y\in S$ and $Ty=x$. Moreover, y is expressible as

$$y=\sum_{j=0}^{\infty} L^{2j}(-\Delta x)$$

which, since $L\Delta=\Delta L$, implies

$$(-\Delta)^k y=\sum_{j=0}^{\infty} L^{2j}(-\Delta)^{k+1}x$$

is non-negative for every $k\geqq 0$. Therefore, y is completely monotone, and since $y\leqq x$, y decreases to 0. Iterating the above argument, we see that $x\in\mathscr{S}$. For a classical proof that completely monotone sequences are moment sequences, see [23].

7. Uniqueness

In the present section and the one which follows we shall take up the question of the uniqueness or non-uniqueness of the measures which occur in (2.13) and (3.15).

In Section 6 we have seen that the cone of all Stieltjes moment sequences (1.1) is $\mathscr{S}$ for an appropriate choice of Σ. Since a measure on $(0, \infty)$ is generally not determined by its Stieltjes moment sequence, there can be no hope that (2.13) and (3.15) are *always* unique representations.

A useful approach to the uniqueness question is via the map $\pi: E \to \Sigma^+, \Sigma^+$ the group of positive characters on Σ, which assigns to each $e \in E$ the character to which it belongs. (E is the set of normalized extremals in (2.13).) For suppose Σ^+ is equipped with a T_2 topology with respect to which π is continuous. If μ is a measure as in (2.13), define a measure μ_0 on the Borel sets of Σ^+ by $\mu_0(A) = \mu(\pi^{-1}A)$. Using the fact μ is finite and supported on a set $\cup_{n=1}^{\infty} \Omega_n$, Ω_n compact metrizable, we obtain from, for example, [16] that there exists for μ_0 almost all $\lambda \in \Sigma^+$ a probability measure μ_λ on the Borel subsets of $\pi^{-1}\lambda$ in $(\cup \Omega_n$, and hence) E such that if $A \subseteq E$ is Borel set, then $F_A(\lambda) = \mu_\lambda(A \cap \pi^{-1}\lambda)$ is integrable with respect to the completion of μ_0, and

$$\mu(A) = \int_{\Sigma^+} F_A(\lambda)\mu_0(d\lambda). \tag{7.1}$$

If V is weakly sequentially complete, then for μ_0 almost all λ there is defined an element $w_\lambda \in \mathscr{S}(\lambda)$,

$$w_\lambda = \int_{E(\lambda)} e\mu_\lambda(de) \tag{7.2}$$

and (2.13) implies

$$w = \int_{\Sigma^+} w_\lambda \mu_0(d_\lambda). \tag{7.3}$$

If $\phi \in V^*$, define a *Laplace transform* of the finite measure $\phi(w_\lambda)\mu_0(d\lambda)$ on Σ^+ by

$$\begin{aligned} \Phi(\sigma, w) &= \phi(\sigma w) \\ &= \int_{\Sigma^+} \lambda(\sigma)\phi(w_\lambda)\mu_0(d\lambda). \end{aligned} \tag{7.4}$$

DEFINITION 7.5. We say $w \in \mathscr{S}$ has the *uniqueness property* if for every $\phi \in V^*$ the measure $\phi(w_\lambda)\mu_0(d\lambda)$ is uniquely determined by its Laplace transform. $\mathscr{S}$ has the uniqueness property if each of its elements does.

Suppose now that $w \in \mathscr{S}$ has a second representing measure $\bar{\mu}$, and let $\bar{\mu}_0$ and $\bar{w}_\lambda$ correspond to $\bar{\mu}$ as μ_0 and w_λ correspond to μ. If ψ is a distinguished element of V^* such that $\psi(e) = 1$, $e \in E$, then $\psi(w_\lambda)\mu_0(d\lambda) = \mu_0(d\lambda)$ and $\psi(\bar{w}_\lambda)\bar{\mu}_0(d\lambda) = \bar{\mu}_0$. Therefore, if w has the uniqueness property, $\mu_0 = \bar{\mu}_0$. Moreover, for arbitrary $\phi \in V^*$ it will be true that $\phi(w_\lambda)\mu_0(d\lambda) = \phi(\bar{w}_\lambda)\bar{\mu}_0(d\lambda)$ when w has the uniqueness property. In all of our examples V^* is separable in the weak-* topology, and therefore there exists a countable set $\phi_1, \phi_2, \cdots$ in V^* such that if $\phi_n(u) = \phi_n(u_0)$, all n, $u, u_0 \in V$, $u = u_0$. It follows in this case that $w_\lambda = \bar{w}_\lambda$ a.e. μ_0. When this is so, $\mu = \bar{\mu}$.

LEMMA 7.6. *Let* $(\Sigma, \mathscr{S})$ *be as in Section* 3, *particularly Theorem* 3.14. *If the cone* $\mathscr{S}(\lambda)$ *is given the natural ordering, then every pair* $f, g \in \mathscr{S}(\lambda)$ *has a greatest lower bound with respect to this ordering. That is,* $\mathscr{S}(\lambda)$ *is a lattice in its natural ordering.*

PROOF. We begin by replacing each $K \in \Sigma$ by $K/\lambda(K) = K_\lambda$ so that $K_\lambda v = v$, $v \in \mathscr{S}(\lambda)$. Thus, we may and shall assume $\lambda = 1$. Given $f, g \in \mathscr{S}$ define $f \wedge g$ to be the greatest lower bound of f and g in the space of ν-measurable functions. The set $\{K(f \wedge g) \mid K \in \Sigma\}$ has a lower bound (for example, 0) in the space of ν-measurable functions and therefore it has a greatest lower bound h (refer to [4]). Since $K(f \wedge g) L\, Kf \wedge Kg = f \wedge g$, $K \in \Sigma$, we have for all K, $K' \in \Sigma$,

$$\begin{aligned} h &\leqq K \circ K'(f \leqq g) = K' \circ K \ (f \wedge g) \\ &\leqq K'(f \wedge g) \end{aligned}$$

and since K' is arbitrary, $h \leqq Kh \leqq h$. Thus $h \in \mathscr{S}(1)$, and clearly h is a greatest lower bound for f and g.

A basic result in the Choquet theory is that when a cone such as $\mathscr{S}(\lambda)$ is a lattice with respect to its natural ordering, then each $w_\lambda \in S(\lambda)$ is represented by a unique measure μ_λ on $E(\lambda)$. See [15, Chap. 9]. Combining this with the fact that V^* (defined preceding Lemma 3.8) is separable, we obtain the following theorem.

THEOREM 7.7. *Let* $(\Sigma, \mathscr{S})$ *be as in Section* 3 *and particularly Theorem* 3.14. *If* $f \in \mathscr{S}$ *has the uniqueness property* (*Definition* 7.5), *then* f *admits a unique representation* (3.15).

EXAMPLE 7.8. Suppose Σ is cyclic, say $\Sigma = \{K^n\}$. If $w \in \mathscr{S}$ is such that for each $\phi \in V^*$ the sequence $\phi(K^n w)$ is uniquely represented as a Stieltjes moment sequence, then w has the uniqueness property. Thus, if for example $K^n w$ is a bounded subset of V, w has the uniqueness property. This justifies the remark made in Section 5 to the effect that uniqueness prevails in the representations of the elements of $\mathscr{S}_\infty$.

Consider next the case of an abelian convolution semigroup on a group $G = LK$ as in Section 4. Since L is normal, there is for each $k \in K$ an automorphism $l \to k^{-1}lk$ of L. This automorphism induces a natural automorphism α_k on L^+, and if μ and $p_c(\cdot)$ are as in the representation (4.10), we define μ_k on L^+ by $\mu_k = p_c(k)\alpha_k\mu$. Then for fixed $k \in K$ the representation (4.10) reduces to a Laplace transform

$$f(lk) = \int_{L^+} c(l)\mu_k(dl) \tag{4.10'}$$

of μ_k. If L, as an abstract group, is *divisible* (for every $x \in L$ and $n > 0$ there exists $y \in L$ such that $y^n = x$), then a measure on L^+ is uniquely determined by its Laplace transform (assuming the latter is everywhere finite). (Refer to [19, p. 504].) Thus, if L is a divisible group, the representation (4.10) is unique for every $f \in \mathscr{S}$.

8. Convolution operators on a semi-simple Lie group

Let G be a connected, noncompact, semi-simple Lie group with finite center, and let $G = KAN$ be an Iwasawa decomposition for G. Let Σ be an abelian convolution semigroup as in Section 4 with the additional assumption that Σ contains at least one element which is left invariant under K. As was proved in Section 4, $\mathscr{S}$ in this case is independent of Σ and consists of all functions f which are transforms of measures on $\mathscr{C} \times X$ (see Section 4 for definitions),

$$f(g) = \int_{\mathscr{C} \times X} \sigma_\lambda(g, x)\mu(d(\lambda, x)),$$

and this will be seen to imply uniqueness of the representation (4.32). We shall eventually prove that μ is uniquely determined by f.

Let (,) be a positive definite inner product on $\mathfrak{a}$, for example, the Killing form, and let $\| \lambda \| = (\lambda, \lambda)^{\frac{1}{2}}$, $\lambda \in \mathfrak{a}$.

LEMMA 8.1. *Let ϕ be a non-negative, compactly supported, measurable function on G, and suppose there is a neighborhood U of e and an $\varepsilon > 0$ such that $\phi \geqq \varepsilon$ on U. There exist numbers $\alpha = \alpha(\varepsilon, U) > 0$ and $\beta = \beta(\varepsilon, U) > 0$ such that*

$$\int_G \phi(h^{-1})\sigma_\lambda(h,x)dh \geqq \alpha \exp(\beta \| \lambda \|) \tag{8.2}$$

for all $\lambda \in \mathfrak{a}$, $x \in X$.

PROOF. We may assume, by decreasing U if necessary, that $U = U^{-1}$ and $KU = U$ (because K is compact). It will be sufficient to obtain the bound (8.2) for $\phi = \varepsilon\chi_U$, χ_U the characteristic function of U. For this choice of ϕ the left side of (8.2) is independent of $x \in X$ (because $\phi(kg) = \phi(g)$, $k \in K$, K is transitive on X, and p_λ is a K-multiplier). Thus we may suppose $x = x_0$ (the identity coset of $P = MAN$).

Identify $\mathfrak{a}$ and $\mathfrak{a}^*$ by means of $(\, , \,)$. There exists a neighborhood Y of e in G and a $\gamma > 0$ such that if $V = \{\lambda \mid \| \lambda \| < \gamma\}$, then

(a) $\exp(V) \subseteq U$,
(b) if $\| v \| = \gamma$, $a = \exp(v)$, then $Ya \subseteq U$, and
(c) if $\| v \| = \gamma$, $y \in Y$, then $\| H(ya) - v \| < 1/2 \| v \|$ $(a = \exp(v))$.

Fix $\lambda \in \mathfrak{a}$, and define $v_\lambda = -(\gamma/\| \lambda \|)\lambda$. If $a_\lambda = \exp(v_\lambda)$, and if $y \in Y$, then $(\lambda, H(ya_\lambda)) = (\lambda, v_\lambda) + (\lambda, H(ya_\lambda) - v_\lambda) \leqq -\gamma \| \lambda \| + (1/2)\gamma \| \lambda \| = -(1/2)\gamma \| \lambda \|$.

Therefore, if $m = \inf_{u \in U} \exp -(\rho, H(u))$, it follows that

$$\begin{aligned}\int_G \phi(h^{-1})p_\lambda(h,x_0)dh &\geqq \varepsilon \int_U \exp -(\lambda + \rho, H(h))dh \\ &\geqq \varepsilon m \exp(1/2)\gamma \| \lambda \|).\end{aligned}$$

Setting $\alpha = \varepsilon m$ and $\beta = (1/2)\gamma$, the lemma is proved.

LEMMA 8.3. *With notation as above, if* $f \in \mathscr{S}$ *is represented by a measure* μ *on* $\mathfrak{a}^* \times X$, *the integrals* $\int_{\mathfrak{a}^* \times X} \exp(M \| \lambda \|)\mu(d(\lambda, x))$ *are finite for every* $M < \infty$.

PROOF. Fix $\phi \in \Sigma$. A simple change of variable shows that if $\varepsilon > 0$ and a neighborhood U of e are such that $\phi \geqq \varepsilon$ on U, and if $\phi^{(n)}$ is the n-fold convolution of ϕ, then

$$\int_G \phi^{(n)}(h^{-1})\sigma_\lambda(h,x)dh \geqq \alpha^n \exp n\beta \| \lambda \|$$

where $\alpha = \alpha(\varepsilon, U)$ and $\beta = \beta(\varepsilon, U)$ are as in the preceding lemma. Another simple computation shows that for all n,

$$\infty > \phi^{(n)} * f(e) \geqq \int_{\mathfrak{a}^* \times X} \phi^{(n)} * \sigma_\lambda(e,x)\mu(d(\lambda,x))$$

$$\geqq \alpha^n \int_{\mathfrak{a}^* \times X} (\exp n\beta \| \lambda \|)\mu(d(\lambda,x))$$

and the lemma is proved.

Suppose μ is a measure on $\mathfrak{a}^* \times X$ such that

$$f(g) = \int_{\mathfrak{a}^* \times X} \sigma_\lambda(g,x)\mu(d(\lambda,x)) \tag{8.4}$$

is finite for every $g \in G$. Of course μ can be replaced by a measure supported on $\mathscr{C} \times X$, also representing f, and we shall later so restrict μ. Let μ_0 be the image of μ on $\mathfrak{a}^*$ under the first coordinate map of $\mathfrak{a}^* \times X$. Arguing, as in Section 7, there exists for μ_0 almost every $\lambda \in \mathfrak{a}^*$ a probability measure μ_λ on X such that $\mu = \int_{\mathfrak{a}^*} \mu_\lambda \mu_0(d\lambda)$, the notation being obvious. Define f_λ, $\lambda \in \mathfrak{a}^*$, by

$$f_\lambda(g) = \int_X \sigma_\lambda(g,x)\mu_\lambda(dx). \tag{8.5}$$

(Of course $f_\lambda(\cdot)$ is defined only for μ_0 almost all λ.)

In what follows we identify $\mathfrak{a}^*$ and $\mathfrak{a}$ by means of the Killing form $(\ ,\ )$. Thus if $\lambda \in \mathfrak{a}$, $\lambda(\cdot) = (\cdot, \lambda)$. Let $\| \cdot \|$ be the corresponding norm. From the definition of p_λ it is evident that for each $g \in G$ there exist constants A_g, $B_g < \infty$ such that for all λ and x,

$$\sigma_\lambda(g^{-1},x) \leqq A_g \exp(B_g \| \lambda \|). \tag{8.6}$$

Notice also that by Lemma 8.3 the measure μ_0 has the property that for every $M < \infty$,

$$\int_{\mathfrak{a}^*} \exp(M \| \lambda \|)\mu_0(d\lambda) < \infty. \tag{8.7}$$

In particular, if $P(\cdot)$ is any polynomial on $\mathfrak{a}$,

$$\int_{\mathfrak{a}} | P(\lambda) | f_\lambda(g^{-1})\mu_0(d\lambda) < \infty \qquad (g \in G). \tag{8.8}$$

Let $D(G,K)$ be the algebra of differential operators on G which are left-invariant under G and right-invariant under K. $D(G,K)$ can be identified with the algebra of G-invariant differential operators on the symmetric space G/K. (See [8, Ch. X].)

According to Karpelevič ([11, Sect. 15 and 17]) there exists for every $D \in D(G,K)$ a polynomial $P_D(\cdot)$ on $\mathfrak{a}$ such that for all λ and x,

$$D\sigma_\lambda(g^{-1},x) = P_D(\lambda)\sigma_\lambda(g^{-1},x) \tag{8.9}$$

holds on G. From (8.9) and the definitions it follows that

$$Df_\lambda(g^{-1}) = P_D(\lambda)f_\lambda(g^{-1}) \qquad (D \in D(G/K)). \tag{8.10}$$

Next, (8.8), (8.10), and an exhaustion of $\mathfrak{a}$ by a sequence of relatively compact open sets imply that

$$Df(g^{-1}) = \int_{\mathfrak{a}} P_D(\lambda)f_\lambda(g^{-1})\mu_0(d\lambda) \tag{8.11}$$

holds in the weak sense for all $D \in D(G,K)$. Applying this in particular to the operators $D = D_0^n$, $n \geqq 1$, where D_0 corresponds to the (elliptic) Laplace-Beltrami operator on G/K we obtain from the continuity of the right-hand side of (8.11) and the regularity theorem ([1, Chap. 6]) that f is C^∞ and (8.11) holds strongly or all D. (Regard $F(g)=f(g^{-1})$ as a function on G/K to apply the regularity theorem.)

REMARK. It should be pointed out that Karpelevič uses an Iwasawa decomposition $G = NAK$ with respect to which he defines functions $P(x,\xi,\lambda)$, on (in the present notation) $G \times X \times \mathfrak{a}$, where if $\xi = k\xi_0$, $\xi_0 \sim P$ in X, $p(x,\xi,\lambda) = \exp(\rho + \lambda, \log a_1)$, where $k^{-1}x = n_1a_1k_1$. Since then $x^{-1}k = k_1^{-1}\,a_1^{-1}\,n_1^{-1}$ and $\log a_1^{-1} = -\log a_1$, we see that $p(x,\xi,\lambda) = \sigma_\lambda(x^{-1},k\xi_0)$ in our notation. It is the functions $p(\,\cdot\,,\xi,\lambda)$ which Karpelevič proves have the properties asserted here for $\sigma_\lambda(\,\cdot\,,\xi)$. These facts also follow readily from the discussion in [8, Chap. X].

Since we have now identified $\mathfrak{a}$ and $\mathfrak{a}^*$, we shall write $w^t\lambda$ for the action of w in the Weyl group on λ if λ is regarded as being in $\mathfrak{a}^*$. From [11] and [8] we also know that if $P_D(\,\cdot\,)$ is as in (8.9),

$$P_D(w^t\lambda) = P_D(\lambda) \qquad (\lambda \in \mathfrak{a}).$$

Moreover, if $P(\,\cdot\,)$ is a W-invariant polynomial on $\mathfrak{a}$, there exists $D \in D(G,K)$ such that $P = P_D$.

By (8.6) we know that $f_\lambda(g^{-1}) \leqq A_g\exp(B_g\|\lambda\|)$, and therefore (8.7) implies that for each $g \in G$ the measure $f_\lambda(g^{-1})\mu_0(d\lambda)$ has an everywhere finite Laplace transform

$$\mathscr{L}(f,g^{-1},y) = \int_{\mathfrak{a}} \exp(\lambda,y)f_\lambda(g^{-1})\mu_0(d\lambda). \tag{8.12}$$

Define for each $n \geqq 0$ a polynomial $Q_n(y,\,\cdot\,)$ by

$$Q_n(y,\lambda) = \frac{(\lambda,y)^n}{n!}$$

and note that $Q_n(wy,\lambda) = Q_n(y, w^t\lambda)$. By the dominated convergence theorem and (8.6)–(8.7) we have $\mathscr{L} = \Sigma_{n=0}\mathscr{L}_n$, where

$$\mathscr{L}_n(f, g^{-1}, y) = \int_{\mathfrak{a}} Q_n(y,\lambda) f_\lambda(g^{-1})\mu_0(d\lambda). \tag{8.13}$$

Let $P_n(y,\lambda) = \Sigma_{w \in W} Q_n(wy,\lambda)$. Clearly $P_n(y, w^t\lambda) = P_n(wy,\lambda) = P_n(y,\lambda)$, $w \in W$ and therefore, by the result stated above, there exists for each n an operator $D_n = D_n(y) \in D(G,K)$ such that $P_n = P_{D_n}$. Applying this to (8.13) and (8.11) we find for all n

$$\sum_{w\in W} \mathscr{L}_n(f, g^{-1}, wy) = D_n f(g^{-1})$$

and therefore

$$\sum_{w\in W} \mathscr{L}(f, g^{-1}, wy) = \sum_{n=0}^{\infty} D_n f(g^{-1}) \tag{8.14}$$

the sum converging. The left side of (8.14) can be rewritten as

$$\begin{aligned} \sum_{w\in W} \mathscr{L}(f, g^{-1}, wy) &= \int_{\mathfrak{a}} \sum_{w\in W} \exp(w^t\lambda, y) f_\lambda(g^{-1})\mu_0(d\lambda) \\ &= \int_{\mathfrak{a}} \exp(\lambda, y) \sum_{w\in W} w^t\{f_\lambda(g^{-1})\mu_0(d\lambda)\} \end{aligned} \tag{8.15}$$

where by $w^t\{f_\lambda(g^{-1})\mu_0(d\lambda)\}$ we mean the image of the measure in braces under w^t.

Now we make the crucial assumption that μ is supported on $\mathscr{C} \times X$, and therefore μ_0 is supported on $\mathscr{C}$. It is well known that for any $w \in W$, w^t acts as the identity on $w^t\mathscr{C} \cap \mathscr{C}$, and therefore knowledge of $\Sigma_{w\in W} w^t\{f_\lambda(g^{-1})\mu_0(d\lambda)\}$ implies knowledge of $\{f_\lambda(g^{-1})\mu_0(d\lambda)\}$. Now by (8.14), the left side of (8.15) depends only upon the given function f, and therefore by the uniqueness theorem for the bilateral Laplace transform the measure $\Sigma_{w\in W} w^t\{f_\lambda(g^{-1})\mu_0(d\lambda)\}$ depends only upon f. By the above remark it follows that the measure $f_\lambda(g^{-1})\mu_0(d_\lambda)$ depends only upon f. Setting $g = e$ it follows that μ_0 is uniquely determined by f. Letting g run through a countable dense set and then using the continuity of $f_\lambda(\cdot)$, we obtain that for almost all λ, $f_\lambda(\cdot)$ is determined by f. Finally, as mentioned in connection with (4.28), f_λ uniquely determines μ_λ, and therefore we have that f determines not only μ_0 but also μ_λ for μ_0 almost all λ. In other words, f determines μ. We now prove the following.

THEOREM 8.16. *Let G be a connected, noncompact, semi-simple Lie group with finite center, and let K, A, N, X, $\mathfrak{a}^*$, and $\sigma_\lambda(\cdot,\cdot)$ have their previous* *meanings. Suppose for each $\lambda \in \mathscr{C}$ there is given a positive element q_λ in $C(X)$ such that for all $g \in G$ the function $q_\lambda(gx)/q_\lambda(x)$ is Borel measurable on $\mathscr{C} \times X$. If a Borel measure μ on $\mathscr{C} \times X$ has an everywhere finite transform*

$$f(g) = \int_{\mathscr{C}\times X} \sigma_\lambda(g,x)\,\frac{q_\lambda(gx)}{q_\lambda(x)}\,\mu\,(d(\lambda.x))$$

then μ is uniquely determined by f and $\{\, q \,|\in \lambda\, \mathscr{C} \}$.

PROOF. We have only to deal with the case in which not all q_λ's are constant. To this end, define α_λ, $\lambda \in \mathscr{C}$, by $\alpha_\lambda = \int_K q_\lambda(kx)dk$ (any x) and $f_K(g) = \int_K f(kg)dk$. Then f_K is a transform

$$f_K(g) = \int_{\mathscr{C}\times X} q_\lambda(g,x)\mu_0(d(\lambda,x)) \tag{8.17}$$

where $\mu(d(\lambda,x)) = (\alpha_\lambda/q_\lambda(x))\mu(d(\lambda,x))$. We know a measure μ_0 is determined by (8.17), and therefore if $\bar{\mu}$ is a second representing measure for f, and if $\bar{\mu}_0$ is similarly related to $\bar{\mu}$, then $\mu_0 = \bar{\mu}_0$. But this certainly implies $\mu = \bar{\mu}$, and the theorem is proved.

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Department of Mathematics
Rice University
Houston, Texas, U. S. A.